done. As a first example we consider a generator with a Gaussian distribution:

$$P(x) = e^{-(x-x_0)^2/2\sigma^2}. \tag{B.7}$$

From the central limit theorem, which states that the sum of many uncorrelated random numbers is distributed according to a Gaussian with a width proportional to $1/\sqrt{N}$, it follows directly that we can obtain a Gaussian distribution just by adding n uniform random numbers; the higher n the more accurate this distribution will match the Gaussian form. If we want to have a Gaussian with a width σ and a centre $\bar{x}$, we transform the sum S of n uniform random numbers according to

$$x = \bar{x} + \sigma S\sqrt{3.0/n}. \tag{B.8}$$

This method for achieving a Gaussian distribution of the random numbers is very inefficient as we have to generate n uniform random numbers to obtain a single Gaussian one. We shall discuss more efficient methods below.

More generally, one can make a nonuniform random number generator using a real function f and for each number x generated by a uniform generator, taking $f(x)$ as the new nonuniform random number, where f is a function designed such as to arrive at the prescribed distribution P. As the number of random numbers lying between x and $x+dx$ is proportional to dx and the same number of nonuniform random numbers $y = f(x)$ will lie between y and $y+dy$ with $dy/dx = f'(x)$, the density of the numbers $y = f(x)$ is given by $1/f'(x)$, so this should be equal to the prescribed distribution $P(y)$:

$$1/f'(x) = P(y) \quad \text{with} \tag{B.9a}$$

$$y = f(x). \tag{B.9b}$$

We must construct a function f which yields the prescribed distribution P, i.e. which satisfies Eq. (B.9b). To this end, we use the following relation between f and its inverse f^{-1}:

$$\left(f^{-1}\right)'(y) f'(x) = 1 \tag{B.10}$$

from which it follows that

$$P(y) = \left(f^{-1}\right)'(y). \tag{B.11}$$

There is a restricted number of distributions for which such a function f can be found, because it is not always possible to find an invertible primitive function to the distribution P. A good example for which this *is* possible, is the Maxwell distribution for the velocities in two dimensions. Taking the Boltzmann factor $1/(k_B T)$ equal to $1/2$ for simplicity, the velocities are distributed according to

$$P(v_x, v_y)dv_x dv_y = e^{-v^2/2}dv_x dv_y = e^{-v^2/2}v dv d\varphi = P(v)dv d\varphi, \qquad \text{(B.12)}$$

so the norm v of the velocity is distributed according to

$$P(y) = ye^{-y^2/2}. \qquad \text{(B.13)}$$

From (B.11) we find that the function f is defined by

$$f^{-1}(y) = -e^{-y^2/2} + \text{Const.} = x \qquad \text{(B.14)}$$

so that

$$y = f(x) = \sqrt{-2\ln(\text{Const.} - x)}. \qquad \text{(B.15)}$$

Because x lies between 0 and 1, and y between 0 and ∞, we find for the constant the value 1, and a substitution $x \to 1 - x$ (preserving the interval $[0,1]$ of allowed values for x) enables us to write

$$y = f(x) = \sqrt{-2\ln(x)}. \qquad \text{(B.16)}$$

The method of generating random numbers by having a function f acting on uniform ones is very efficient since each random number generated by the uniform generator yields a nonuniformly distributed random number.

From (B.12), we see that it is possible to generate Gaussian random numbers starting from a distribution (B.13), since we can consider the Maxwell distribution as a distribution for the generation of two independent Gaussian random numbers $v_x = v\cos\varphi$, $v_y = v\sin\varphi$. From this it is clear that by generating two random numbers, one being the value v with a distribution according to (B.13) and another being the value φ with a uniform distribution between 0 and 2π, we can construct two numbers v_x and v_y which are both distributed according to a Gaussian. This is called the *Box–Müller method.*

If we cannot find a primitive function for P, we must use a different method. A method by Von Neumann uses at least two uniform random numbers to generate a single nonuniform one. Suppose we want to have a distribution $P(x)$ for x lying between a and b. We start by constructing a generator whose distribution h satisfies $h(x) > \alpha P(x)$ on the interval $[a,b]$. A simple choice is of course the uniform generator, but it is efficient to have a function h with a shape roughly similar to that of P. Now for every x generated by the h-generator, we generate a second random number y uniformly between 0 and 1 and check if y is