Glenn Thierens
2nd Master Civil Engineering
Dr. Prof. Ir. C. Hiel

Homework 2: Innovation of construction materials

Problem 1

For the polygon, using notations as in the assignment, the shape factor is to be determined.

$$\phi = 12\frac{I}{A^2}$$

In theory, this value results from a ratio of moments (elastic and plastic).

In order to determine ϕ, the area moment of inertia and the area have to be determined.

1. Area moment of inertia

In order to find the area moment of inertia, we start from a base triangle which constitutes the polygon by rotation around the center point.

For a rectangular triangle, by elaborating

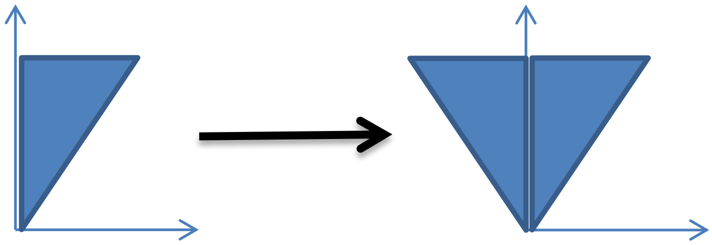
$$I_{x,\Delta} = \int y^2 dy = \int_0^r y^3 \tan\frac{\pi}{r} dr = \frac{r^4}{4}\tan\frac{\pi}{n}$$

It follows that

$$I_x = \frac{bh^3}{36}$$

$$I_y = \frac{hb^3}{12}$$

These values can be doubled to obtain the value for a base triangle due to symmetry



$$I_x = \frac{bh^3}{18}$$

$$I_y = \frac{hb^3}{6}$$

From this:

$$I_z = I_x + I_y$$

Using Steiner, the moment of inertia is calculated with respect to the central axis:

$$I' = I + A \cdot d^2$$

$$I = \left(\frac{bh^3}{18} + \frac{hb^3}{6}\right) + \frac{4}{9}h^3 b$$

That value has to be divided by two in order to have an in-planar moment of inertia.

$$\phi = \frac{12I}{A^2} = \frac{6\left(\frac{bh^3}{2} + \frac{hb^3}{6}\right)n}{n^2 b^2 h^2}$$

Using

$$A = b \cdot h$$

$$b = \sin\frac{\pi}{n}$$

$$h = \cos\frac{\pi}{n}$$

One finally obtains:

$$\phi = \frac{3\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}}{n\sin\frac{\pi}{n}\cos\frac{\pi}{n}}$$

The same can be done using Maple:

$$A := \frac{n}{2} \cdot a^2 \cdot \sin\left(\frac{2 \cdot \mathrm{Pi}}{n}\right)$$

$$\frac{1}{2} n a^2 \sin\left(\frac{2\pi}{n}\right)$$

$$Ix := \frac{A}{6} \cdot a^2 \cdot \left(1 + \frac{1}{2} \cdot \cos\left(\frac{2 \cdot \mathrm{Pi}}{n}\right)\right)$$

$$\frac{1}{12} n a^4 \sin\left(\frac{2\pi}{n}\right)\left(1 + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right)\right)$$

$$\phi_x := \frac{12 \cdot Ix}{A^2}$$

$$\frac{4\left(1 + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right)\right)}{n \sin\left(\frac{2\pi}{n}\right)}$$

$$\xrightarrow{\text{expand sin(2*Pi/n)}}$$

$$\frac{2\left(1 + \frac{1}{2}\cos\left(\frac{2\pi}{n}\right)\right)}{n \sin\left(\frac{\pi}{n}\right)\cos\left(\frac{\pi}{n}\right)}$$

Goniometric identities allow re-writing this in the shape proposed in the homework assignment.

$$\phi = \frac{2\left(1 + \frac{1}{2}\cos\frac{2\pi}{n}\right)}{n \sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{2 + \cos\frac{2\pi}{n}}{n \sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{2 + \cos^2\frac{\pi}{n} - \sin^2\frac{\pi}{n}}{n \sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{3\cos^2\frac{\pi}{n} + \sin^2\frac{\pi}{n}}{n \sin\frac{\pi}{n}\cos\frac{\pi}{n}}$$

QED

Next, it will be demonstrated that above expression has a true maximum for n=3.

In order to achieve this, ϕ is rewritten:

$$\phi = \frac{2}{n} \cdot \frac{2 + \cos\left(\frac{2\pi}{n}\right)}{\sin\left(\frac{2\pi}{n}\right)}$$

In order to prove the maximum, a series development of the latter part is performed:

$$\phi = \frac{2}{n} \cdot \left(\frac{3\,n}{2\,\pi} + \frac{2\;\pi^3}{15\,n^3} + O\left(\frac{1}{n^5}\right)\right) = \frac{3}{\pi} + \frac{4\;\pi^3}{15\,n^4} + O\left(\frac{1}{n^6}\right)$$

The derivative of this expression is negative for $n \geq 3$ thus $\phi(n)$ is a continuously decreasing function. This means that n is the natural number still representing a polygon, with the largest shape factor.

In conclusion: the shape factor has a true maximum for n=3.

Problem 2

This problem is assessed on a separate piece of paper.