Newton's superb theorem: An elementary geometric proof

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Abstract

Newton's "superb theorem" for the gravitational $1/r^2$ force states that a spherically symmetric mass distribution attracts a body outside as if the entire mass were concentrated at the center. This theorem is crucial for Newton's comparison of the Moon's orbit with terrestrial gravity (the fall of an apple), which is evidence for the $1/r^2$ law. Newton's geometric proof in the *Principia* "must have left its readers in helpless wonder" according to S. Chandrasekhar and J.E. Littlewood. In this paper we give an elementary geometric proof, which is much simpler than Newton's geometric proof and more elementary than proofs using calculus.

I. INTRODUCTION

Newton's "superb theorem" for the gravitational $1/r^2$ force states that a spherically symmetric mass distribution attracts a body outside as if the entire mass were concentrated at the center. The name "superb theorem" was used by S. Chandrasekhar in *Newton's Principia for the Common Reader*[1] and by J. W. L. Glaisher.[2] See also I. B. Cohen and A. Whitman.[3]

The superb theorem is of crucial importance for Newton's comparison of terrestrial gravity (the fall of the apple) with the orbit of the Moon. Newton wrote that[4,5] "The same year [1666] I began to think of gravity extending to the orb of the Moon ... From Kepler's Rule of the periodical times of the Planets ... I deduced that the forces which keep the Planets in their Orbs must [be] reciprocally as the squares of their distances from the centers ...: and thereby compared the force requisite to keep the Moon in her Orb with the force of gravity at the surface of the earth, and found them to answer pretty nearly." The superb theorem is also crucial for an exact solution of the Kepler problem: Finding the orbit of a planet under the attraction of the sun (neglecting the attraction between planets) for two spherically symmetric bodies with a $1/r^2$ force between mass elements.

When Newton first made the comparison of the Moon's orbit with the fall of the apple, he had no knowledge of the superb theorem. He assumed that considering the mass of the Earth to be concentrated at the center to obtain the magnitude of the gravitational field on the surface of the Earth was nothing more than an approximation: "It might be ... accurate enough at greater distances ... wide off the truth near the surface of the planet ..., where the distances of the particles are unequal ...."[6] Newton suspected the superb theorem to be false until 1685, as he wrote in his letter to Halley of 20 June 1686.[7]

Newton proved the superb theorem in 1685, one year before finishing the *Principia*, in which the theorem was published as Proposition 71 of Book I in 1687. We know from Newton's own words that "he had no expectation of so beautiful a result till it emerged from his mathematical investigation."[8] In Chandrasekhar's words: "The superb theorem is most emphatically against common sense ... unless one had known its truth already."[9]

Chandrasekhar translated Newton's geometrical proof into modern notation.[10] See also Refs. 11 and 12. Chandrasekhar reported the comment by Littlewood that Newton's geometrical construction for the proof "must have left its readers in helpless wonder."[13,14] D.T.

Whiteside characterized Newton's proof as "opaque and overlong."[15]

In this paper we give a geometric proof of the superb theorem, which is much simpler than Newton's geometric proof. Our geometric proof has four elementary steps and is also simpler than the proofs using calculus in textbooks (see, for example, Refs. 16 and 17). Our proof is suitable for introductory physics courses without calculus and is accessible to good high school students. In typical graduate textbooks on classical mechanics the superb theorem is not mentioned.[18]

In Sec. II we give Newton's elementary geometric proof for a test mass *inside* a spherical mass-shell. The geometry of this proof is closely related to the geometry of our new proof in Sec. IV for Newton's superb theorem, that is, for a test mass *outside* a spherical mass-shell. In Sec. III we discuss the role of infinitesimals in the method of Newton's geometric proofs, a method that is a reformulation of "the manner of ancient geometers," in particular of Archimedes. In Sec. IV A we convert the problem of the superb theorem to an equivalent problem with the geometry of *one point inside a spherical shell*: calculating the spherical average of the radial accelerations at all observation points at a distance r from the center due to only one source point. In Sec. IV B we give the elementary geometric solution of this equivalent problem and hence the proof of Newton's superb theorem. The geometry of the solution of the equivalent problem is much simpler than the geometry of Newton's derivation. The solution of the equivalent problem is a geometric derivation of Gauss's law in integral form for a closed spherical surface without using vector calculus. In Sec. V we give two more propositions of Newton based on the superb theorem (relevant for the falling apple and for the system Moon-Earth), comment on Littlewood's proof of the superb theorem, and note that the solution of the equivalent problem, Gauss's law in integral form for Newtonian gravity, is the same as the $R^0_{\ 0}$ Ricci-tensor equation in integral form of Einstein's gravity (general relativity) for sources that are nonrelativistic relative to the observer.

II. NEWTON'S GEOMETRIC PROOF FOR A TEST MASS WITHIN A SPHERICAL MASS SHELL

For a test mass inside a homogeneous spherical shell of matter, Newton's Proposition 70 of Book I states: "If to every point of a spherical surface there tend equal centripetal forces decreasing as the square of the distances from those points, I say, that a corpuscule placed within that surface will not be attracted by those forces any way."[19]

For the analogous case of the $1/r^2$ law of electrostatics, Newton's proposition 70 was tested by the null experiment of Henry Cavendish in 1773.[20]

Newton's elementary geometric proof of Proposition 70 (Ref. 19) is shown in Fig. 1: With the test body at the observation point P as the apex, construct any cone of infinitesimally small solid angle $d\Omega$ that intersects the spherical source-shell in both directions by the source-surface areas dS and dS' around the source points Q and Q'. The attractions per unit test mass at P by these two areas, dS and dS', are equal and opposite because (1) the gravitational force decreases as the square of the distance, while the surface areas of the source, cut out by the cone of given solid angle $d\Omega$, increase as the square of the distance; and (2) the angles θ between the infinitesimally thin cone (in both directions) and the normals to the source sphere at both intersection points (radial lines OQ and OQ') are equal. Because the entire solid angle around the point P can be divided into such double cones, the resultant attraction is zero. Q.E.D.

III. INFINITESIMALS IN NEWTON'S GEOMETRIC PROOFS

Newton's geometric proof of Proposition 70 (our Sec. II) uses his "evanescent quantities," that is, vanishing quantities (infinitesimals), and in particular his "method of ultimate ratios" (title of Sec. I of Book I):[21] In the Scholium at the end of this section Newton explains " . . . the ultimate ratio of evanescent quantities is to be understood not as the ratio of quantities before they vanish . . . , but the ratio with which they vanish." Newton emphasizes that "ultimate ratios with which quantities vanish are not ratios of ultimate quantities, but limits with which the ratios of quantities decreasing without limit are continually approaching, and which they can approach so closely that their difference is less than any given quantity."[22] In the case of the proof of Prop. 70, it is the ultimate ratio of the magnitudes of the attraction by the evanescent (that is, infinitesimal) source areas. Newton does not write (dS, dS'), instead he discusses the analogous arcs in his plane figure and in his proof. In his notation he uses the endpoints of the small arcs. But Newton emphasizes that his evanescent arcs can be replaced by straight lines: Newton's evanescent arcs are infinitesimal.

Throughout the Principia, Newton uses ultimate ratios, with which he is "solidly in a contemporary tradition of Fermat, Blaise Pascal, Huygens, James Gregory, and Isaac

Barrow."[23] In this scholium Newton wrote that "I have presented these lemmas ... to avoid the tedium of ... lengthy proofs by 'reductio ad absurdum' in the manner of the ancient geometers."[24]

Euclid and Archimedes presented early forms of infinitesimal thinking in the "method of exhaustion" (of Eudoxus) combined with "reductio ad absurdum."[25–27] An example is the proof by Archimedes that the area inside a circle is equal to the product of the radius times half of the circumference: In the method of exhaustion one can cut the circular disc radially into an increasing number N of identical slices, one compares the thin slices with inscribed (respectively, circumscribed) triangles; that is, one compares the circular disc with an inscribed (respectively, circumscribed) N-sided polygon. N can be chosen arbitrarily large. In a simplification by Leonardo da Vinci one considers N even and reassembles the triangles in an alternating sequence to obtain an area arbitrarily close to a rectangle for N arbitrarily large. If somebody claimed that the area of the circular disc is some given number which is smaller than the radius times half the circumference, one "reduces this to absurdum" (proof by contradiction) by choosing N so large that the total area of the inscribed triangles is larger than the claimed result. One then repeats the argument with circumscribed triangles.

Newton's geometric proofs in the Principia use evanescent quantities (i.e. infinitesimals). But these proofs do not use the machinery of calculus developed by Newton and Leibniz.

IV. THE ELEMENTARY GEOMETRIC PROOF OF THE SUPERB THEOREM

In Sec. IV A we convert the problem of the superb theorem (Proposition 71, a test mass outside a homogeneous spherical shell) to an equivalent problem using spherical symmetry. In Sec. IV B we give the elementary geometric solution of the equivalent problem using a method similar to the method used by Newton for Proposition 70 (our Sec. II).

A. Spherical symmetry: Translation to an equivalent problem

The superb theorem considers a spherical shell of matter as the source of gravity and gives the (radial) acceleration of a test particle at the observation point P outside the source shell (observation point at a distance r from the center), see Fig. 2(a). Instead we analyze the equivalent problem of one source particle located at Q, as shown in Fig. 2(b) and find the radial components of the accelerations averaged over a spherical observation surface outside the source particle (observation surface at a distance r from the center), as shown in Fig. 2(b). The steps needed for calculating this spherical average are explained before Eqs. (3) and (4).

The proof of the equivalence of the two problems is given in two steps:

1. We consider the average over the spherical observation surface of the radial component of the acceleration, $g_{\text{radial}} = \mathbf{g} \cdot \mathbf{r}/r$, measured outside the spherical mass shell at the distance r from the center. This average is obtained by multiplying g_{radial} by the area element dS on the observation surface, summing the contributions, and then dividing by the total surface area. The average is equal to the radial acceleration at any one of the observation points, such as the point P, because of the spherical symmetry of the source in the first problem, as shown in Fig. 2(a). This first step appears trivial, but it is necessary to make the second step possible.

2. The spherical average on the observation sphere of the radial component of the acceleration yields the same contribution from each single source element of equal mass in the spherical source shell, such as Q, because of the spherical symmetry of the observation surface [see Fig. 2(b)]. Hence, it is sufficient to consider the contribution of only one source mass-element at the point Q, that is, the equivalent problem of Fig. 2(b), and later sum over the equal contributions of all other source elements of equal mass.

The crucial point is that the geometry is much simpler for the solution of our equivalent problem, Fig. 2(b) (one point inside a shell), than the complicated geometry needed by Newton to solve the original problem, Fig. 2(a) (one point outside a shell).

These steps complete the proof of the equivalence of the two problems. The equivalent problem and its solution in Sec. IV B is Gauss's law in integral form for a spherical surface.

B. Geometric solution of the equivalent problem

The equivalent problem of Fig. 2(b) can be solved with the elementary geometric method shown in Fig. 3; the method is analogous to Newtons's method to prove proposition 70 (given in Sec. II). We use the source point Q as the apex and consider an arbitrary ray which starts at Q, goes in one direction, and hits the spherical surface at the observation point P. At P, the magnitude of the gravitational acceleration g is $g_P = Gm/L_{QP}^2$, where m is the small (infinitesimal) source mass at Q, and G is Newton's constant. The angle θ is the angle between the ray QP and the radial direction OP. The radial component of the gravitational acceleration at P is

$$g_{\text{radial}}(P) = -(Gm/L_{QP}^2)\cos\theta. \tag{1}$$

Around this ray from Q to P, we consider a cone starting at the apex Q and of infinitesimally small solid angle $d\Omega$, which will intersect the spherical observation surface by the surface area dS given by

$$dS = (L_{QP}^2)d\Omega/\cos\theta. \tag{2}$$

The first step in calculating the average of the radial accelerations over the observation sphere is to form the product of the radial acceleration given by Eq. (1), with the weight (for averaging), which is the surface area element dS (intersected by the cone) given by Eq. (2):

$$g_{\text{radial}}\, dS = -Gm\, d\Omega. \tag{3}$$

Note the two cancellations in $(g_{\text{radial}}\, dS)$, which are the basis for the simplicity of our proof:

1. The inverse square of the distance in the force law, $1/L_{QP}^2$, cancels L_{QP}^2 in the surface area dS for a cone with solid angle $d\Omega$.

2. The ratio $\cos\theta$ of the radial to the total acceleration cancels $1/\cos\theta$ in the surface area dS for a cone with solid angle $d\Omega$.

The second step in calculating the spherical average is to sum $(g_{\text{radial}}\, dS)$ in Eq. (3) over all surface area elements dS, that is, over the entire observation surface. The sum over $d\Omega$ is 4π. The third step in taking the spherical average of g_{radial} is to divide by the total weight for averageing, which is the total area of the spherical observation surface, $4\pi r^2$. We thus obtain

$$\langle g_{\text{radial}}\rangle^{\text{spherical average}} = -Gm/r^2, \tag{4}$$

This result for the spherical average of the radial accelerations is the same as if the single point mass m were placed at the origin.

The conversion back to the original problem, Newton's superb theorem, now follows. For a spherically symmetric source shell, the acceleration is radial, and because it is the same all over the observation sphere, there is no need to take the average over the spherical observation shell. Therefore $\vec{g} = -(\vec{r}/r)GM_{\rm tot}/r^2$ at every observation point outside the source shell. This result completes our geometric proof of Newton's superb theorem.

Equation (4), which is equivalent to Newton's superb theorem, is Gauss's law in integral form[28,29] for a spherical surface. Gauss's law for the flux integral over a closed surface S enclosing the volume V, $\oint_S \mathbf{g} \cdot \mathbf{dS} = -4\pi G M_V$, a concept from vector calculus, is not taught in undergraduate or high school courses on Newtonian mechanics. The superb theorem follows simply from Equation (4), Gauss's law in integral form for a closed spherical surface.

In taking the average of $g_{\rm radial}$ over the observation sphere, we had to sum over all infinitesimal surface area elements dS. This sum is called an "ultimate sum" in the Principia,[30] which corresponds to an integration in today's language. In the integration over $d\Omega$, the integrand is a constant, as given in Eq. (3). Therefore the integration is reduced to calculating the surface of the sphere.

V. COMMENTS

We have given a simple geometric proof of the superb theorem along the lines of Newton's simple geometric proof of Proposition 70.

For a source with spherical symmetry and a radius-dependent density we add up the accelerations outside the source caused by all shells, and conclude that the result is the same as if all the mass was placed at the origin: Newton's Proposition 74, which is needed for the falling apple.

For two dissimilar spheres with different spherical radius-dependent density distributions we add up the contributions of matter shells and conclude that (in Newton's words at the end of Proposition 76) "the whole force with which one of these spheres attracts the other will be inversely proportional to the square of the distance of the centers," which Newton found "very remarkable." This proposition is needed for the attraction between Moon and Earth.

According to Chandrasekhar,[31] "J. E. Littlewood has conjectured (though I do not share in the conjecture . . .) that Newton had perhaps first constructed a proof based on *calculus* which 'we can infer with some possibility what the proof was'." Chandrasekhar gives Littlewood's conjectured proof, which consists of 14 equations. Our geometric proof with four steps is simpler than Littlewood's proof using calculus.

Equation (4) for Newtonian gravity is the same as the $R^0_{\ 0}$ Ricci-tensor equation in integrated form for Einstein gravity (general relativity) for sources with $v \ll c$ relative to the observer. This Einstein equation determines the spherical average of the radial geodesic deviation (that is, of the radial accelerations of free-falling test particles relative to a free-falling observer).

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[20] J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (John Wiley & Sons, New York, 1999), Sec I.2.

[21] Reference 3, Lemma 1, p. 433, and Scholium on pp. 440–443.

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[23] Reference 3, p. 129.

[24] Reference 3, p. 441.

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Figures

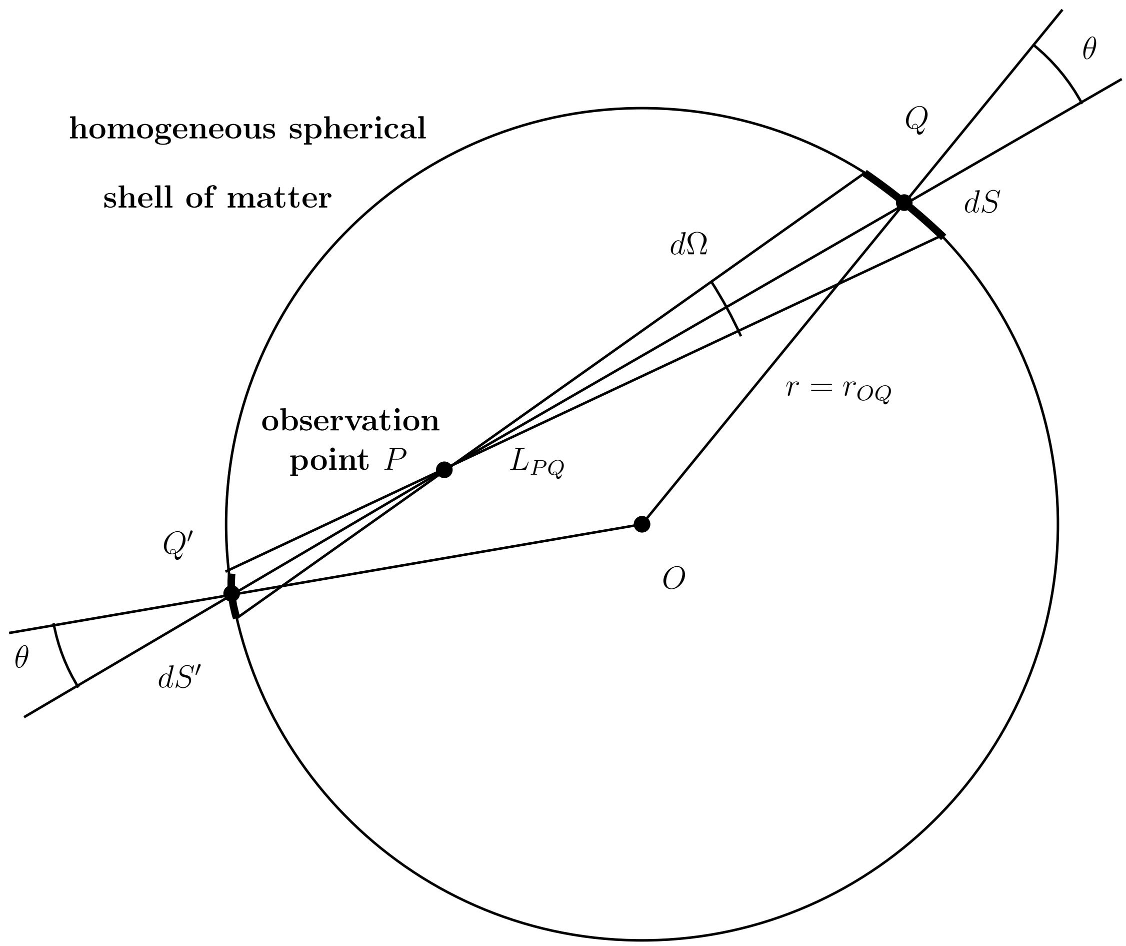

FIG. 1: Newton's proof for a test mass inside a homogeneous spherical shell of matter (Proposition 70). O is the center of a homogeneous spherical shell of matter, P is the position of a test particle, $d\Omega$ is the solid angle of an infinitesimally thin cone with P as the apex, dS, dS' are intersection areas (around Q and Q') of the two sides of the cone with the spherical shell of matter, and θ equals the two equal angles between PQ andPQ' to the normals on the matter shell, OQ and OQ'.

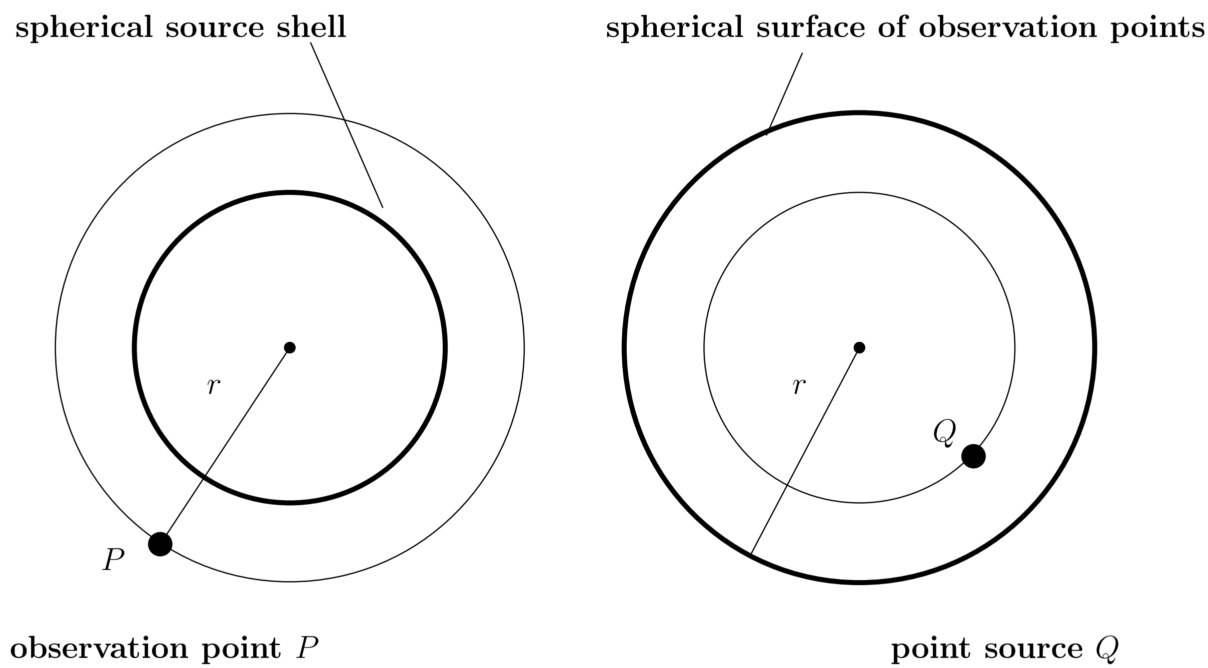

FIG. 2: The two equivalent problems. (a) Spherical source shell (solid inner circle) and test point mass at observation point P outside the source-shell. (b) Spherical observation surface (solid outer circle) and one source point mass at Q inside the observation surface: Our proof for the geometry with one point inside a spherical surface, Fig. 2(b), is much simpler than Newton's proof for the geometry with one point outside a spherical surface, Fig. 2(a).

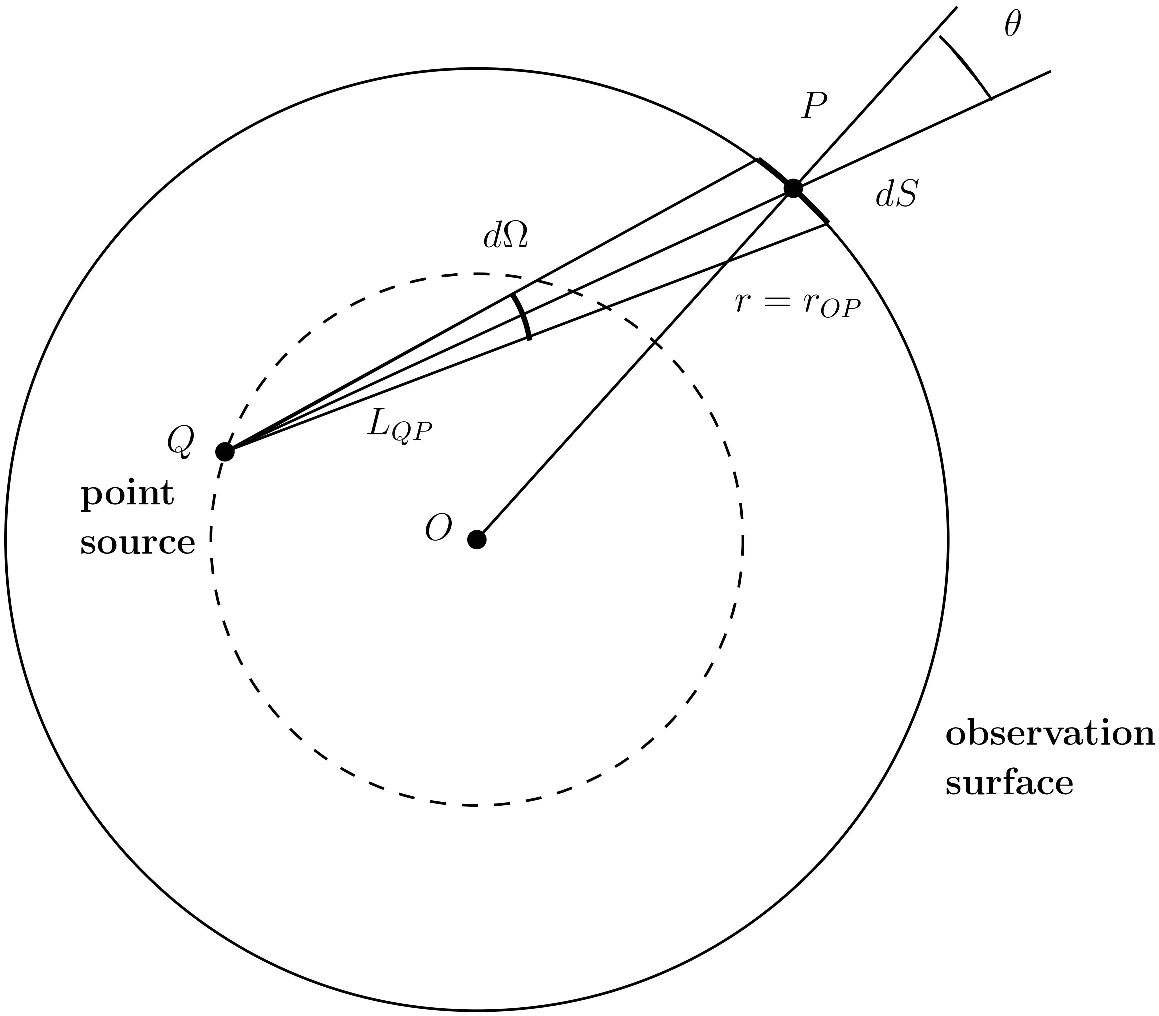

FIG. 3: Elementary proof of Newton's superb theorem (Proposition 71). Q is the source point, P is the observation point, O is the center of the observation surface, θ is the angle between QP and the normal OP on the observation surface, $d\Omega$ is the solid angle of the narrow cone with Q as the vertex, and dS is the surface area of the intersection of the cone with the observation surface.