

PHY481 - Lectures 7 and part of 8

Sections 3.5-3.6 of PS

A. Finding the electric field - continued

(vi) *Uniform shell of charge (shell theorems)*

The shell theorems state that (i) the electric field inside a uniform shell of charge is zero and (ii) that the field outside the uniform shell of charge is the same as that of a point charge with the same total charge as the shell.

These results are easy to derive from Gauss's law, in the same way as for the cylindrical case. However now we use spherical surfaces of radius $r < R$ and $r > R$, where R is now the radius of the spherical shell of charge. The shell has charge density σ and the total charge on the shell is $Q = 4\pi R^2 \sigma$. We first consider a Gaussian surface consisting of a spherical shell, which is concentric with the charged shell, with radius $r < R$. Noting that by symmetry the electric field is directed radially and only depends on r , then since there is no enclosed charge the electric field for $r < R$ is zero in agreement with the first shell theorem. Considering a spherical shell of radius $r > R$, we have,

$$\int_S \vec{E} \cdot d\vec{A} = 4\pi R^2 E(r) = \frac{Q}{\epsilon_0} \text{ so that } \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q\hat{r}}{r^2} \quad (1)$$

which proves the second shell theorem.

Now lets prove the first shell theorem directly. Consider a general point \vec{r} inside the sphere. If we place a spherical co-ordinate system at that point, then we can draw a cone of infinitesimal solid angle $d\Omega = \sin\theta d\theta d\phi$ extending from this origin to the spherical surface. The cone has two intersections with the spherical shell, on opposite sides of the sphere. The areas of the spherical shell subtended by this solid angle are $dA_1 = r_1^2 d\Omega$ and $dA_2 = r_2^2 d\Omega$. The magnitude of the electric field at \vec{r} due to these two areas of charge is given by,

$$dE = k\sigma \left(\frac{dA_1}{r_1^2} - \frac{dA_2}{r_2^2} \right) = 0 \quad (2)$$

This construction applies to all parts of the spherical surface, proving the first shell theorem. Proof of the second shell theorem is more involved.

It is also relatively straightforward to demonstrate the second shell theorem by direct integration, using polar co-ordinates it may be shown that the electric field in the radial direction for $r > R$ where R is the radius of the shell, is given by,

$$E_r(r) = \frac{\sigma}{4\pi\epsilon_0} \int_0^\pi \frac{(r - R\cos\theta) 2\pi R^2 \sin\theta d\theta}{(R^2 \sin^2\theta + (r - R\cos\theta)^2)^{3/2}} \quad (3)$$

which reduces to the point charge result. In deriving this expression we used $dA = 2\pi R^2 \sin\theta d\theta$

(vii) *Uniformly charged sphere of radius R*

Again we carry out the flux integral to find, for a surface of radius r ,

$$\phi_E = E(r)4\pi r^2 \quad (4)$$

The charge enclosed by this surface changes with r . For $r < R$, the enclosed charge is given by,

$$q(r < R) = \frac{4\pi}{3} r^3 \rho \quad (5)$$

while for $r > R$ we have,

$$q(r > R) = \frac{4\pi}{3} R^3 \rho \quad (6)$$

Using Gauss's law $\phi_E = q/\epsilon_0$, we thus have,

$$E(r < R) = \frac{r\rho}{3\epsilon_0} = \frac{Q}{4\pi\epsilon_0} \frac{r}{R^3} \quad (7)$$

where $Q = 4\pi\rho R^3/3$ is the total charge on the sphere. The electric field outside the sphere of charge is like that of a point charge (shell theorem),

$$E(r > R) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \quad (8)$$

In both cases, the direction of the electric field is \hat{r} . (see problem 3.19)

B. Electric potential energy and electric potential

Physical definition

The electric potential energy (U) is the potential energy due to the electrostatic force. As always only differences in potential energy correspond to physical observables. However we define a reference potential energy and calculate all differences in potential energy with respect to this reference. In electrostatics, the potential energy is defined to be zero when the charges are an infinite distance apart.

The difference in potential energy in moving a charge between two positions a and b is defined in terms of the work done in moving the charge between these two positions, so that,

$$\Delta U_{ab} = \int_a^b \vec{F}_{ext} \cdot \vec{ds} = - \int_a^b \vec{F} \cdot \vec{ds} \quad (9)$$

where \vec{F} is the Coulomb force on the charge and \vec{ds} is a small element of distance along the path we choose to take from a to b . Note that Eq. (19) does not depend on the path we take. All paths from a to b give the same change in potential energy. Note also that Eq. (19) is true only for conservative forces, ie. systems in which there is no dissipation (e.g. friction).

Instead of using the Coulomb force, we defined the electric field which is the force per unit charge. In the same way we define the electric potential, V , to be the potential energy per unit charge, ie. $U = qV$. The electric potential is so important it is given its own unit, the volt (V). In terms of the electric potential, equation (19) is,

$$\Delta V_{ab} = - \int_a^b \vec{E} \cdot \vec{ds} \quad (10)$$

From this equation we see that the units of voltage are Nm/C . We have defined the volt to be the unit of voltage, so we have the relation $V = Nm/C$ or $V/m = N/C$. So far we have used N/C as the units of electric field. This is usual in electrostatics. However in circuit theory it is usual to use volts/meter (V/m).

If we move a charge q from a point a to a point b , then the change in potential energy, U_{ab} , of the system is

$$\Delta U_{ab} = - \int_a^b \vec{F} \cdot \vec{ds} \quad (11)$$

- \vec{F} is the electric force on charge q due to all of the other charges in the system
- \vec{ds} is a small displacement along the path we choose to take from a to b .

If we define the electric potential to be the potential energy per unit charge, so that $V = U/q$, then the difference in electric potential is given by,

$$\Delta V_{ab} = \frac{1}{q} \Delta U_{ab} = - \int_a^b \vec{E} \cdot \vec{ds} \quad (12)$$

We define the potential energy and hence the potential of a charge at infinity to be zero.

With the definition of potential given in Eq. (22) and taking the potential at infinity to be zero, we can calculate the potential at each position from the electric field.