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A FORMULA FOR SOLVING GENERAL QUINTICS; A FOUNDATION FOR SOLVING GENERAL POLYNOMIALS OF HIGHER DEGREES

ABSTRACT

I explore possible methods of solving general quintics and higher degree polynomials. In this paper I will attempt to show that each quintic has an auxiliary cubic equation. I will therefore attempt to bring a method of deriving a general quintic and its possible auxiliary cubic equation forms. I propose the same method to be used to generate higher degree polynomials.

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METHOD

Consider the simple cubic equation:

$x^3 = x + 2$ ------------ 1

Squaring both sides of the above equation we obtain the equation:

$x^6 = (x+2)^2 = x^2 + 4x + 4$ -------------- 2

Multiplying both sides of equation 1 by x^3 we obtain the equation:

$x^6 = (x+2)x^3 = x^4 + 2x^3$ ------------- 3

Subtracting equation 2 from 1 and rearranging we obtain the equation:

$x^4 + 2x^3 - x^2 - 4x - 4 = 0$ --------------- 4

Rearranging equation 1:

$x^3 - x - 2 = 0$ ----------------- 5

The cubic equation 5 is an auxiliary equation of the quartic equation 4 since:

$x^4 + 2x^3 - x^2 - 4x - 4 = (x^3 - x - 2)(x - 2) = 0$ ------------ 6

This analysis projects the idea that each polynomial equation possesses auxiliary lower degree polynomial. I will use this concept to come up with a method that can be used to solve any general quintic.

Consider the cubic equation:

$x^3 = (x+a)^2 + bx + c = x^2 + x(b + 2a) + a^2 + c$ -------------- 7

Again consider the cubic equation:

$x^3 = x^2 + dx + e$ --------------- 8

Multiplying equation 8 by 7 we obtain the equation:

$x^6 = x^4 + x^3(b + 2a + d) + x^2(a^2 + c + e + bd + 2ad) + x(dc + be + 2ae + a^2d) + ce$ -----------9

Multiplying both sides of equation 7 by x^3 we obtain the equation:

$x^6 = x^5 + x^4(b + 2a) + x^3(a^2 + c)$ --------------------- 10

Multiplying both sides of equation 8 by x^3 we obtain the equation:

$x^6 = x^5 + x^4d + x^3e$ ----------- 11

Taking equation 10 subtracting equation 9 and rearranging:

$x^5 + x^4(b + 2a - 1) + x^3(a^2 + c - b - 2a - d) - x^2(a^2 + c + e + bd + 2ad) - x\big(dc + be + 2ae + a^2d\big) - ce = 0$ -------- 13

I will call equation 13 an extended quintic equation.

Rewriting the cubic equations 7 and 8:

$x^3 - x^2 - x(b + 2a) - a^2 - c = 0$ --------- 14

$x^3 - x^2 - dx - e = 0$ ----------- 15

The product of equation 7 and 8 is given by:

$[x^3 - \{(x + a)^2 + bx + c\}[x^3 - x^2 - dx - e] = 0$ ------------- 16

Equation 10 can be rewritten as:

$(x^3 + (x + a)^2 + bx + c)x^3 - x^6 = ((x + a)^2 + bx + c)x^3 = 0$ ------------- 17

Adding equation 16 to 17 and simplifying we get the equation:

$x^3(-x^2 - dx - e) - \{(x + a)^2 + bx + c\}(-x^2 - dx - e) = 0$

Or $[x^2 + dx + e][x^3 - \{(x + a)^2 + bx + c\}] = 0$ -------------- 18

The expansion of equation 18 is given by

$x^5 + x^4(d - 1) + x^3(b - 2a - d + e) + x^2(c - a^2 + d(b + 2a) - e) + x[d(c - a^2) + e(b - 2a] + e(c - a^2) = 0$ ------ 19

Equation 19 is a second extended general quintic whose factorized form is equation 18.

The second extended general quintic can be used to obtain a general quintic formula since

$a_4 = d - 1$ -------1 ------------ 19

$a_3 = b - 2a - d + e$ ---------- 20

$a_2 = c - a^2 + d(b + 2a) - e$ ------------- 21

$a_1 = d(c - a^2) + e(b - 2a)$ ------------ 22

$a_0 = e(c - a^2)$ ------------- 23

By equation 19:

$d = a_4 + 1$ -------------- 24

Let $p = c - a^2$ ----------- 25

Let also $q = (b - 2a)$ ---------- 26

Then $a_3 = p - (a_4 + 1\ ) + e$ ------------- 27

$a_2 = p + (a_4 + 1\ )q - e$ -------------- 28

$a_1 = p(a_4 + 1) + eq$------------ 29

Equation 23 can be written as:

$a_0 = -ep$ ------------ 30

Substituting 30 into 26 and simplifying:

$p^2 - p(a_4 + a_3 + 1) + a_0 = 0$ ---------------- 31

$p = \frac{(a_4+a_3+1)\pm\sqrt{(a_4+a_3+1)^2-4a_0}}{2}$ ------------- 31

By equation 30:

$e = \frac{-a_0}{p}$ --------------- 33

By equation 28:

$q = \frac{a_2+e-p}{a_4+1}$ --------- 34

To determine b and c, choose a and then calculate b and c using the equations 25 and 26

Example 1

Solve $x^5 - x + 1 = 0$

Solution

$a_4 = 0 \Rightarrow d = 1$

$p = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$

Take $p = \frac{1+i\sqrt{3}}{2}$

$e = \frac{2}{1+i\sqrt{3}}$

$q = e - p = \frac{3-i\sqrt{3}}{1+i\sqrt{3}}$

Take a =1; $c = p + 1 = \frac{3+i\sqrt{3}}{2}$

$b = q + 2 = \frac{5+i\sqrt{3}}{1+i\sqrt{3}}$

The auxiliary equations of the quintic are:

$x^2 + x + \frac{2}{1+i\sqrt{3}} = 0$

$x^3 - x^2 + \frac{3-i\sqrt{3}}{1+i\sqrt{3}}x + \frac{1+i\sqrt{3}}{2} = 0$

The above two quadratic and cubic equations can be solved by their respective formulae.

The roots of the auxiliary quadratic equation are:

$x_{1,2} = \frac{-1}{2} \pm \sqrt{\frac{-7+i\sqrt{3}}{4(1+i\sqrt{3}}}$

One of the roots of the auxiliary cubic equation is:

$$x_3 = \sqrt[3]{\frac{4}{27} + \sqrt{\frac{16}{729} + \frac{1}{27}(\frac{-3+i\sqrt{3}}{3(1+i\sqrt{3})})^3}} - \sqrt[3]{\frac{4}{27} + \sqrt{\frac{16}{729} + \frac{1}{27}(\frac{-3+i\sqrt{3}}{3(1+i\sqrt{3})})^2}} + \frac{1}{3}$$

CONCLUSION

A general formula for solving quintics is achievable. Irreducible quintics can be solved by the formula method proposed. Given a_0, a_1, a_2, a_3 and a_4 one can determine a, b, c, d, e and hence solve the quintic equation.

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