

Gravitational Torsion Balance

Spencer Tomarken, Richard Helms, Med Webster, and Will Johns

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1 Measurement of G

1.1 Background

Suggested Reading: Ref. 1, pp. 75-92.

In his work published in 1686, *Principia Mathematica*, Isaac Newton¹ theorized that all masses exert a gravitational force on each other, and that the force is proportional to the masses of the two objects and inversely proportional to the square of the distance between them. Newton's theory explained both the bodies falling on Earth and the motion of the planets. His law allowed mathematicians to correctly predict the existence of the planet Neptune, after astronomers noticed an irregularity in the orbit of the planet Uranus. Today's form of Newton's law of gravitation is expressed as

$$F = \frac{GmM}{r^2}, \quad (1)$$

but neither the gravitational constant G nor any of the astronomical masses was known until over a century after Newton. Consequently, all calculations in Newton's time were performed by using the ratio of the mass of either the Sun or the Earth to that of the planet or satellite. Newton's actual equation was of the form

$$F = \frac{4\pi^2 Km}{r^2}, \quad (2)$$

where m is the mass of the object, r is the distance, and K is Kepler's constant, which is the ratio of the cube of the radius of a planet's orbit to the square of the period of revolution[2].

In 1798 Henry Cavendish² set up an experiment[3] to measure the mass of the Earth. To accomplish this feat, he first found the value for G in Eq. 1. He then used the known value for the radius of the Earth to calculate the Earth's gravitational force on an object of known mass. Note that G was not measured until 110 years after Newton's law of gravitation was well established. The measurement of G is still difficult; most of the fundamental constants such as e and h have been measured to eight or nine significant figures, but G has been measured[4] only to four:

$$G = (6.67428 \pm 0.00067) \times 10^{-11} \text{ m}^3/\text{kg s}^2 \quad (3)$$

An review[5] of recent techniques is available. The speed of propagation of the gravitational field is even more difficult to measure and the first measurements of the speed were rumored early in 2003, but we have not seen publication of the work.

The gravitational force law, like the electrostatic force law, is inversely proportional to the square of the distance between two objects. Consequently Gauss's law applies to gravitational attraction just as it does to electrostatic attraction; the force between spheres of uniform density is the same as the force between equal point masses at the centers of the spheres. However, measurements of G in mines suggest a departure from inverse square at the level of a few standard deviations, and searches for departures from inverse square, particularly at small distances, is an active field of research.

¹Sir Isaac Newton, 1642-1727, English polymath. When not inventing differential and integral calculus as well as explaining universal gravitation and studying light, he was Warden of the Mint.

²Henry Cavendish, 1731-1810, English chemist and physicist; discovered hydrogen, as well as argon (without realizing it), and the composition of water and nitric acid.

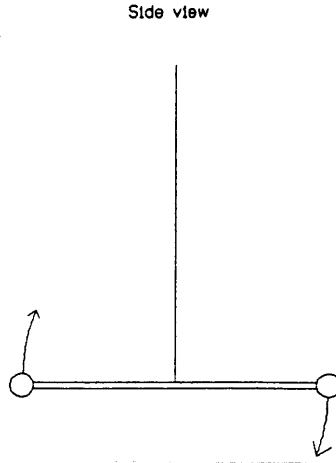


Figure 1: Schematic view of torsion balance motion. The dumbbell of small balls on a crossbar is suspended on a very thin wire. The oscillation is about the wire as an axis.

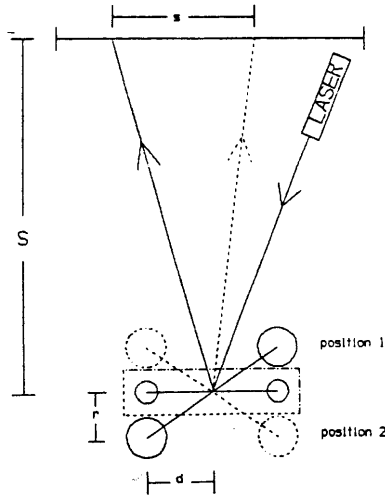


Figure 2: Top view of the Cavendish experiment apparatus showing the two position of the large balls. Note that a rotation of the mirror produces an angular displacement of the spot which is twice as large.

The torsion balance remains an important research instrument. A new limit on the photon mass, less than 10^{-51} grams or 7×10^{-19} electron volts, has been established by an experiment[6] in which light is aimed at one side of a sensitive torsion balance; if light had mass, the rotating balance would suffer an additional tiny torque. This represents a 20-fold improvement over previous limits on photon mass. Photon mass is expected to be zero by most physicists, but this is an assumption which must be checked experimentally. A nonzero mass would make trouble for special relativity and Coulomb's inverse-square law for electrical force and would thus modify Maxwell's equations. These same authors have also carried out a measurement[7] of the universal gravitational constant, G , and are currently measuring the force of gravity at the sub-millimeter range (a departure from Newton's inverse-square law might suggest the existence of extra spatial dimensions) and are studying the Casimir force, a quantum effect in which nearby parallel plates are drawn together.

1.2 Cavendish's Method

To determine G , Cavendish used a torsion³ balance. The torsion fiber supports a freely oscillating horizontal bar holding two small lead balls as in Fig. 1. Two larger balls on a support arm can be positioned next to the small balls in either of the two positions shown in Fig. 2. Thus the equilibrium position about which the balance oscillates is different for the two positions and it is this difference in the equilibrium positions which we use to determine G . The angular displacement is measured by reflecting a light from a mirror attached to the torsion balance onto a distant scale as in Fig. 2. But this requires knowledge of the torsion constant of the fiber. The period of the oscillations and the mass of the small balls could be measured to determine the torsion constant of the fiber, but elimination of the torsion constant from the equation for the angular displacement also eliminates the mass of the small balls as shown below.

The torque exerted by both larger balls on the smaller ones is

$$\tau = 2Fd \quad (4)$$

where τ is the torque, F is the force of one of the large balls on its small neighbor, and d is the length of the support arm. If the large balls start in one extreme position and are moved to the other, the resulting change in angle of the balance is α , which causes a 2α rotation of the reflected light beam. (See Fig. 1.) The rest position of the balance without external attraction is halfway between the two extremes, so each of these positions is an $\frac{\alpha}{2}$ rotation away from its rest position. The torque is related to the torsion spring constant, κ , as

$$\tau = \frac{1}{2}\kappa\alpha \quad (5)$$

The torsion spring constant can be determined empirically from the period of the undamped oscillations using the formula for the period of undamped, rotational oscillation

$$T_o = 2\pi\sqrt{\frac{I}{\kappa}} \quad (6)$$

or equivalently,

$$\kappa = \frac{4\pi^2 I}{T_o^2}, \quad (7)$$

where I is the moment of inertia of the balance. Since the small lead balls weigh considerably more than the aluminum bar which supports them, one can approximate the moment of inertia as just that of the balls, or

$$I = 2md^2 \quad (8)$$

where m is the mass of each small ball, and d is the length of the arm supporting each small ball. Combining Eqs. 4-8 gives us the equation

$$F = \frac{\pi^2 \alpha 2md}{T_o^2} \quad (9)$$

The angle α is the rotation of the mirror and is related to the displacement of the laser spot, s , and the perpendicular distance from the mirror to the scale, S , by the relation

$$2\alpha = \tan^{-1}\left(\frac{s}{S}\right) \quad (10)$$

or in the small angle approximation,

$$2\alpha = \frac{s}{S} \quad (11)$$

Substituting Eq. 11 into Eq. 9 and setting it equal to the force in Eq. 1, with M as the mass of each large ball, we obtain an expression for G in terms of measurable quantities.

$$G = \frac{\pi^2 dsr^2}{MST_o^2} = \frac{dsr^2\omega_o^2}{4MS} \quad (12)$$

Note that not only the torsion constant but also the masses of the small balls have been eliminated.

³torsion comes from the Latin verb **torquēre** , to twist, whence the word torque

1.3 Improved Treatment with Variation of Gravitational Torque

The derivation in the previous section treats the gravitational torque as a constant which simply displaces the equilibrium position:

$$\kappa\theta_o = \frac{2GMmd}{r^2}, \quad (13)$$

and the torsion proportionality constant is related to the undamped angular frequency of the balance by $\kappa = I\omega_o^2$. The force is not constant because the distance between large and small balls changes as the balance rotates and the observed balance period is modified by both damping and by the variation of the force between the balls. The variation of the period is, in fact, the basis of recent measurements[7] of G .

The geometry is shown in more detail in Fig. 3. The displacement with the front large ball on the right is θ_o , the positive direction of θ is the direction of increasing numbers on the meter stick used to measure the spot position, and ω_o is the undamped angular frequency of the balance with no large balls present.

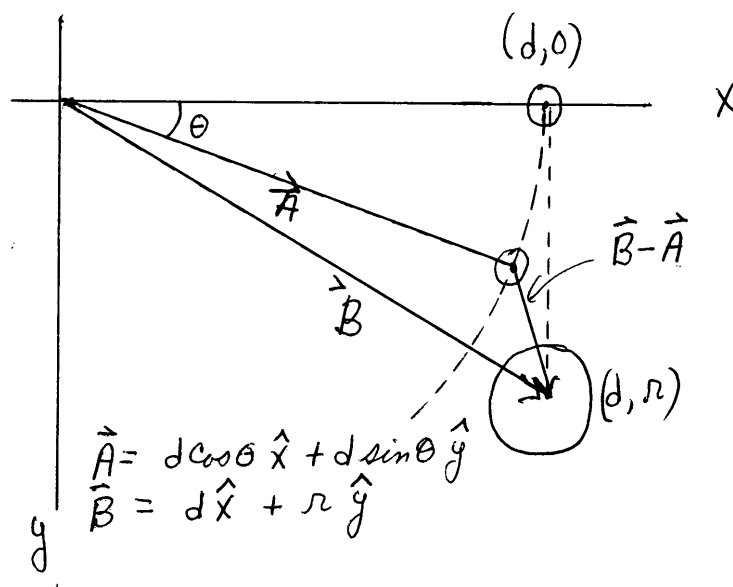


Figure 3: Top view of the right half of the Cavendish balance. The equilibrium position of the small ball would be on the x-axis if the large ball was removed, and the balance is shown rotated through a positive angle, θ . The x and y -axes are as shown and the z -axis is into the paper, as required for a right handed coordinate system. The position of the small ball is shown by the vector \mathbf{A} which has components $(d \cos \theta, d \sin \theta)$, the vector \mathbf{B} to the large ball has components (d, r) , and the direction of the force on the small ball is the vector $\mathbf{B} - \mathbf{A}$.

As the balance rotates, each small ball moves closer to and farther from its adjacent large ball and the direction of the force changes. Both of these effects must be included in the expression for the torque. A considerable amount of geometry and trigonometry is most easily accomplished by writing Newton's law of gravity in vector form and calculating the torque as the cross product of the line from the axis to the position of the small ball with the force. This notation is shown in Fig. 3 and the vector form of the gravitational force on the small ball is

$$\mathbf{F} = GMm \frac{\mathbf{B} - \mathbf{A}}{|\mathbf{B} - \mathbf{A}|^3}. \quad (14)$$

For a torsion balance, it is the torque which is needed, and the torque is the cross product of the vector from the axis to the point of application of the force with the force vector. In the notation of this problem,

$\tau = \mathbf{A} \times \mathbf{F}$. Since the direction of \mathbf{F} is $\mathbf{B} - \mathbf{A}$, we start by working out $\mathbf{A} \times (\mathbf{B} - \mathbf{A})$. All three of these vectors lie in the x - y plane, so the cross product has only a z component,

$$\mathbf{A} \times (\mathbf{B} - \mathbf{A}) = \mathbf{A} \times \mathbf{B} - \mathbf{A} \times \mathbf{A} = (A_x B_y - A_y B_x) \hat{\mathbf{z}} = (d \cos \theta r - d \sin \theta d) \hat{\mathbf{z}} \quad (15)$$

$$|\mathbf{B} - \mathbf{A}| = \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2} = \sqrt{(d - d \cos \theta)^2 + (r - d \sin \theta)^2} \quad (16)$$

Multiplying by 2 to include the torque on the other small ball and factoring out a rd in the numerator and a r^3 in the denominator, the more accurate replacement for Eq. 4 for the gravitational torque on the balance is

$$\tau_z = \frac{2GmMd}{r^2} \frac{\cos \theta - \frac{d}{r} \sin \theta}{\left[1 + 2 \left(\frac{d}{r}\right)^2 (1 - \cos \theta - \frac{r}{d} \sin \theta)\right]^{3/2}} \quad (17)$$

The region of small oscillations is best explored by looking at the series expansion of the torque as a series expansion in θ .

$$\tau_z = \frac{2GmMd}{r^2} \left[1 + \left(2\frac{d}{r}\right) \theta + \left(6\left(\frac{d}{r}\right)^2 - 1\right) \frac{\theta^2}{2} \right] \quad (18)$$

The first term just gives the displacement of the equilibrium position as in our original discussion. The linear term combines with the torsion constant, modifying the period. The quadratic leads to a nonlinear distortion from the simple damped oscillation. The effects of the quadratic term are negligible and are discussed in the appendix.

The torque which has been calculated is due to the large ball which is near the small ball. There is also a torque due to the attraction of each small ball to the opposite large ball. Because the separation of this pair is larger and the force is not perpendicular to the lever arm, it is much smaller. Consequently we treat it as a constant and write the magnitude of the torque as $\frac{2GmMd}{r^2} \cdot C$ where C is a dimensionless constant which is much less than one and depends only upon the geometry of the instrument. Calculation of the value of C is left to the student in item 4 of the results section.

With this expression for the gravitational torque, the differential equation (Newton's second law) becomes

$$I \frac{d^2 \theta}{dt^2} + \beta \frac{d\theta}{dt} + \left(\kappa - \frac{4d^2 GmM}{r^3} \right) \theta = \frac{2dGmM}{r^2} \left[1 + C + \left(6 \left(\frac{d}{r} \right)^2 - 1 \right) \frac{\theta^2}{2} \right]. \quad (19)$$

If the quadratic term is negligible, this is just the usual differential equation for damped harmonic motion except that the effective spring constant has a more complicated form. The resonant frequency of a simple damped oscillator is decreased slightly, $\omega^2 = \omega_o^2 - \lambda^2$ where $\lambda = \beta/2I$ is the measured damping constant. Combining the damping shift with the effective change in the torsion constant, the angular frequency which is observed with the large balls in place is given by

$$\omega_m^2 = \frac{\kappa}{I} - \frac{4d^2 GmM}{I r^3} - \lambda^2 \quad (20)$$

. Note that both terms which modify the frequency in Eq. 20 are negative so that the measured angular frequency is slightly less than ω_o , the angular frequency with no external balls and no damping.

The solution to the differential equation is

$$\theta(t) = \theta_\infty + \theta_{\max} e^{-\lambda t} \cos(\omega_m t + \phi). \quad (21)$$

where θ_{\max} and ϕ are dependent on the position and angular velocity at $t=0$ and θ_∞ is the angular displacement due to the large balls, the effect we seek to measure. Note that θ_∞ is not a constant of integration but is determined by the differential equation: For $t \gg 1/\lambda$, $\theta(t) \Rightarrow \theta_\infty$ and all derivatives approach zero, so for large t the differential equation gives a relationship between G and θ_∞ .

$$\left(\kappa - \frac{4d^2 GmM}{r^3} \right) \theta_\infty = \frac{2dGmM}{r^2} [1 + C]. \quad (22)$$

2 Apparatus

wiimote
green laser
tripod torsion balance Leybold No.332 10
ruler with mounted LEDs

The torsion balance is very delicate and must be treated with care. The balance arm is suspended on a very fine wire from a pin through the knob at the top of the shaft. The knob can be rotated to adjust the equilibrium position correctly. The knob is secured by two screws which come through the side of the mount and into a groove in the knob. Underneath the knob is a spring which pushes upward on the knob. If both of these clamping screws are removed, the spring will push the knob upward and break the support wire. The screw with the slotted head is not tight but simply insures that the shaft cannot move upward even if the screw with the large head is removed (by mistake). The screw with the large head clamps the pin so that it does not rotate and may be loosened slightly to adjust the equilibrium position and then tightened to make measurements. You should not have occasion to adjust the screw with the slotted head. There is a stop in the groove so that the suspension can be rotated through slightly less than a full circle, and that should be more than adequate for your adjustments. Passing the stop requires great care that one of the screws is always in the groove, or the spring will push the mount upward and destroy the fine suspension wire. Since the instructor has to replace broken suspension wires, it is only fair that you request his advice and assistance for any adjustments of the slotted head screw.

In the experiment a green laser is reflected off a mirror attached to the balance arm onto a ruler mounted on an adjacent wall. To more easily measure the angle of deflection of the apparatus, the position of the laser on the wall is measured. The laser's position and the elapsed time previously were measured by hand while the balance was oscillating for thirty or forty minutes. In order to increase the precision of the measurement as well as collect more data points, you will use a computer program to measure the laser's position digitally with a wiimote. The wiimote is a handheld remote control device designed for use with the Wii video game system. It is equipped with accelerometers, infrared light sensors, and Bluetooth capabilities. In addition to reducing the error in the experiment this method also greatly reduces the burden formerly placed on the experimenter. Measuring the data by hand used to be a very lengthy and tedious process which took hours to complete. By using the wiimote the tedium of taking data by hand has been eliminated.

Because the laser's position is being measured by a remote control with its own coordinate system, two LEDs are mounted on the ruler to serve as reference points. The program reports the horizontal displacement from the left LED in meters. In addition the LEDs help to serve as guides during the calibration process. They will be displayed on the computer and the experimenter will adjust the wiimote until the LEDs are within view. It is okay if the LEDs are not exactly parallel with the x -axis or if the laser's reflection is not perfectly collinear with the LEDs because the computer program uses trigonometry to measure the horizontal distance between the left LED and the laser.

3 Procedure

3.1 Setup

1. Wear a pair of latex gloves whenever handling the lead balls.
2. Using the mounting apparatus, secure the wiimote firmly in the upper clamp and direct the wiimote's infrared sensors towards the ruler on the wall. While it is turned off, mount the laser in the lower clamp pointed towards the torsion balance. On the torsion balance apparatus, there are two large black plastic screws on the underside of the rectangular case. These screws secure the small balls in the casing. Unscrew these so that the torsion balance can hang freely. It will take at least ten to twenty minutes for the balance to settle down to a steady oscillatory motion. It will most likely be bouncing off the glass casing of the apparatus and the spot will be bouncing vertically as well as sweeping horizontally when you first release it. Do not take data while there

is significant vertical bobbing. Ask the instructor to explain why and to back up his comments on coupled oscillations by demonstrating the Wilberforce Pendulum.

3. You will have to adjust the laser so that it reflects off the torsion balance's mirror and strikes the ruler mounted on the wall. Use caution to avoid staring into the laser beam or pointing it at eye-level. While the laser is intended for routine laboratory use, it can nonetheless be harmful if used irresponsibly. You will see two reflected spots on the wall, the stable one is from the glass case and the erratic one is from the mirror. The reflection from the glass is a good measure of the rotational orientation of the balance's casing. Flip the switch at the end of the meter stick to turn on the LEDs. These LEDs are at 15 cm and 85 cm on the meter stick. The program which communicates with the wiimote uses the wiimote output for these two reference spots and the laser spot to compute the distance in meters from the led at 15 cm.
4. On the computer sign in as **student** with the password given to you by your instructor. Click the terminal icon located on the toolbar (it's a small black box that resembles a TV). Type **WiIDAQ** into the command line and press enter. It will request you to push buttons 1 and 2 simultaneously on the wiimote to activate its Bluetooth protocol. You will have to press the buttons within about five seconds of pressing enter or else you will have to repeat the command. After doing so, the program will start and you will be brought to a menu.
5. Check the battery indicator at the top of the screen. If it indicates less than about 30% lifetime remaining, replace the batteries in the wiimote. The wiimote has to be removed from the clamp to replace the batteries, and cannot be reclamped with sufficient precision to continue a measurement. It is necessary to restart the experiment if any of the elements is moved.
6. Choose the first option and begin the setup process. Two plots will appear on the screen. The top plot is of the x and y -coordinates of the LEDs and the reflection of the laser spot. Adjust the wiimote so that the LEDs fill the width of the screen and are nearly parallel with the x -axis. Verify that the laser's reflection is nearly collinear with the LEDs. Watch the laser's reflection for one or two full periods to verify that its equilibrium is within 10 cm of the center of the ruler. If not, adjust the black metal knob on the top of the balance. A very small rotation results in a large displacement and it will take several minutes to take effect.
7. Next you need to "pump up" the balance's amplitude. By placing the larger lead balls in the support arm and positioning them so that their gravitational attraction increases the balance arm's momentum you can increase the small lead balls' amplitude. You should rotate the arm until the large lead balls are just touching the glass casing in order to exert the maximum gravitational force. Be careful not to slam the balls into the glass casing. Besides the risk of breaking the equipment, you might also disturb the experiment by causing the mirror to shift uncharacteristically. Use the lower plot to determine when to rotate the large balls. They should be rotated every half period or when the displacement curve on the lower plot reaches a minimum or maximum (velocity is zero). Continue until the amplitude of the motion of the spot is between 10 and 15 cm peak-to-peak. Now leave the large lead balls in one position or the other. (You will record the other one next.) Once everything is set, press **Esc** on the terminal to go to the menu.
8. Next you must determine the number of sample points per minute that you want to collect. Because of the speed of the spot, the default number will be set at 6 samples per minute, which will record the laser's position on the ruler once every 10 seconds. You may change this if you feel that one data point every 10 seconds is too fast or slow.
9. Next, you must name the file to which the data will be written. The default name will be **rawData.txt**. It is important that you save your data to a different file each time you run the experiment. If a file name already exists, the program will warn you and ask if you want to overwrite the old file. Do so with caution and make sure that you are not deleting your own data.

3.2 Recording Data

1. Choose option 4 to start taking data. You should record at least five full oscillations. While collecting data the same two plots will appear but the raw data which will be used to find the equilibrium shift will be written to the previously specified file. Observe the plots to make sure that nothing goes wrong.
2. Once you have observed five full oscillations, press **Esc** on the terminal to stop the data collection. Select 1 so that you can observe the spot position more easily as you increase the amplitude as you did previously. Be sure to remember to stop the large balls in the alternate position from before. Press **Esc** to quit the calibration process. Select the sample rate desired (if different) and choose a new file name.
3. Now choose 4 to begin collecting data. Record the data for another five full oscillations.
4. Once you have completed this second session, if time permits, repeat the measurements for the first session. This is to ensure that your results were valid and that the apparatus is consistent and was not bumped or tampered with.
5. Once you are done, press **Esc** on the terminal, and type 5 to quit the program. Close the terminal.
6. Save all of your work and make copies of the files that you can keep (i.e. on a USB thumb drive). You should be able to find your file through the Places menu on the toolbar or you can enter **ls -ltr** to get a listing of the files in the directory. Do not count on your work being saved on the laboratory computer.
7. Place everything back as you found it and be sure to turn off the laser, LEDs, and the wiimote to preserve battery power.
8. Wash your hands if you handled the lead balls.

3.3 Plotting and Fitting

The goal of this experiment is to determine G . Through collecting this data you want to find the shift in the equilibrium position of the torsion balance arm in order to gauge the force of gravity exerted by the large lead balls on the smaller ones. In order to measure the shift in equilibrium position you must find fit curves for the data that you just collected. The fit curves will be damped sinusoidal functions of this form: $f(t) = Ae^{-\lambda t} \cos(\omega t + \phi) + y_o$. A is the amplitude, λ is the damping constant, ω is the angular frequency, ϕ is the phase, and y_o is the offset. And of course t is time, the independent variable. To fit the data the computer returns values for these constants. In order to fit the data sets you must include estimated guesses for the various parameters in the equations. These initial guesses for the variables give the computer starting points to fit the curves. If you collected a third set of data you can use both the first and second data sets to find the offset and then substitute the third data set for the first and find a second offset to verify the consistency of the experiment. Average these offset values together. If your two offsets do not agree, you must repeat the experiment until they are consistent. You will use a plotting and fitting program, gnuplot, to extract the relevant displacement from your data. Typing **gnuplot** in a terminal will bring up the program in a new window. You will need to know two gnuplot commands, **plot** and **load**. The plot command followed by a space and the name of a WiiDAQ output file in quotes (" ") will plot the data in the file. The load command followed by the name of a script file in quotes will execute the commands in the script. The files **noballs** and **nobally** were taken with no large balls on the apparatus in order to explore the effect of large oscillations. You can plot them and fit them with the script file noballs.gp.

1. Open a new terminal. Type **gedit plot.gp &** to open gedit which is an editing program. This is a powerful text editing program, but the minimal editing you will be doing requires only the facilities to move the cursor, the backspace key to delete text, and then typing in the corrected text. These functions are similar to using Notepad in a windows system.

2. The first step will be to type the file names of the your first and second data sets in quotes next to where it says **File 1** and **File 2**.
3. Further down the script are first guesses at the parameters of the curves to be fitted. The fitter is pretty robust and will usually find the right answer, but it sometimes fails if the starting values for the search are way off. The phase is the usual source of difficulty. Look at plots of your data and correct these starting values. To see plots of your data, go back to the terminal and type **gnuplot**. In the gnuplot window type **plot 'name of your first file'**.
4. After you have entered your starting values, save the file.
5. In the terminal at the gnuplot command line type **load 'plot.gp'**. This will plot your data and fit curves as well as print out the numerical values for the constants of the fit equations. If you have entered accurate enough initial values, you should see a graph with your data plots as well as two fit curves with their equilibrium displacements clearly marked. If you do not see any graph, you have not entered accurate enough initial guesses. Return to gedit and use your data plots to again estimate the constants of the fit equations. It may take three or four runs of estimation before gnuplot is able to produce an accurate fit curve.
6. The two displacements, y_o , given as fit results are in the coordinate system of the Wii output. The zero is over near the left end of the meter stick. The equilibrium position with no large balls present is midway between them, so the difference is twice the displacement we are looking for and does not depend upon the zero of the coordinate system.
7. If you took three files of data, repeat this process but replace the name of the first file with the name of the third file. After you find the new shift in equilibrium, average this with the first value for the equilibrium shift that you found.

4 Results

1. Measure the distance from the balance to the meter stick on the wall. Measure the diameter of one of the large balls and the thickness of the case of the balance, including the glass. Since the small balls are at the center of the case, these two measurements enable you to compute the distance, r , between the large and small balls. The balance itself is not accessible for measurement, so you are given that the radius of the balance arm to the center of the small ball, d , is 5.0048 ± 0.010 cm. The mass of one large lead ball is 1.4987 ± 0.0010 kg.
2. What is S ? Considering that S is the distance from the apparatus to the center of the oscillations on the meter stick, the distance that you measured for S may or may not be accurate depending on whether or not your oscillations occurred at the center of the meter stick. If they were offset to the either the left or right, you will need to recalculate S using the Pythagorean theorem.
3. Determine the measured period of oscillation from your fit curves. Combine or average appropriately.
4. In the derivation we represented the attraction to the other large ball is a smaller force with a component in the opposite direction. Calculate the ratio of the relevant component of the small force to the larger one. This is the ratio called C in the text. What is the sign of C ?
5. Express θ_∞ in terms of the difference in the two displacements you obtained by fitting your data. Be careful with factors of two. θ_∞ is the angular displacement from the no balls equilibrium position. The reflected beam moves through twice the angle which the mirror moves. Use this result and equations 20 and 22 to express G in terms of measured quantities.
6. What is your value of G ? Calculate the error in your measurement from your estimates of the error in the angular frequency (ω), separation of equilibrium points (s), distance from the mirror to the center of oscillations on the scale (S), and the distance between large and small balls (r).

7. Uncertainties in each of the measured quantities contributes to the computed error in G . Which of the measurements would it be most important to improve to reduce the computed error? Hint: In computing the error on your measurement of G in item 6 you added the squares of contributions due to errors in each of the measured quantities. Which made the largest contribution?
8. Cavendish originally performed this experiment to determine the mass of the Earth. If you know the radius of the Earth to be 6370 km and the acceleration due to gravity to be 9.800 m/s^2 , what would your value of G lead you to publish as the value of the mass of the Earth? If you have already done the Kater Reversible Pendulum Experiment, you may wish to use your measured value of g .
9. Why did we pick a balance with such a long period? Why not use a shorter period balance to save time?
10. There is a metal screen on the inside of the glass windows of the balance. Why was it put there? (The appropriate answer will show your understanding of the relative strengths of electrical and gravitational forces in practical situations.)

5 Linearity and Calibration of the Wiimote

In order to verify that the wiimote's measurements are consistent, a linearity test has been conducted. A light source was held at known horizontal points against a ruler while the x -coordinate of the wiimote was read. A graph of the Wiimote horizontal position versus ruler position is shown in Fig. 4.

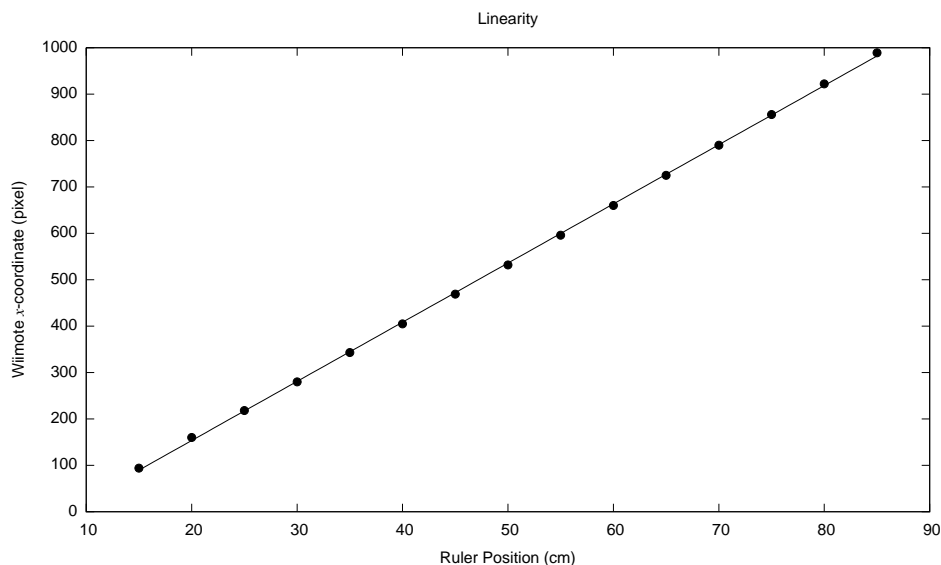


Figure 4: Graph of the Wiimote's x -coordinate versus the known position on a ruler

6 Justification for Neglecting Quadratic Term

In order to estimate the importance of the quadratic term we note that the observed motion is still oscillatory, qualitatively similar to the expected behavior for small oscillations, even for quite large amplitude oscillations (but not so large that the small balls bounce off the glass). Thus we expect that the kinetic energy at the equilibrium position is equal to the potential energy at the next maximum excursion plus a small loss due to damping, and we compare the error in the maximum potential energy due to the reading error with the potential energy which corresponds to the quadratic term. The contribution to

the potential energy from the linear term is

$$PE_{\max} = \int_0^{\theta_{\max}} \kappa \theta \, d\theta = \frac{1}{2} \kappa \theta_{\max}^2. \quad (23)$$

We estimate the error in θ_{\max} from the uncertainty in the wiimote's measurement of the spot on the meter stick on the wall. The wiimote's precision is affected by the amount of pixels allocated to measuring the 70 cm stretch of the meter stick between the two LEDs. To determine this, in your data files, subtract the value in the "Right-x" column from the value in the "Left-x" column. Divide 70 cm by twice this value, and you will find the uncertainty, σ . And note (Eq. 10) that the error in θ_{\max} is $\sigma/2S$. Your exact uncertainty will depend on the distance from the apparatus to the wall as well as the orientation of the wiimote. It should be around 0.3 mrad. The usual propagation of error gives the derivative of the PE times the error in θ_{\max} or $\kappa \times 3 \times 10^{-4}$. The contribution to the potential energy from the quadratic term is

$$PE_{\max} = \int_0^{\theta_{\max}} \frac{2dGmM}{r^2} \left(6 \left(\frac{d}{r} \right)^2 - 1 \right) \theta^2 \, d\theta = \frac{1}{3} \frac{2dGmM}{r^2} \left(6 \left(\frac{d}{r} \right)^2 - 1 \right) \theta_{\max}^3 \quad (24)$$

Equating this PE_{\max} to the error in the PE_{\max} from the linear term, gives $\theta_{\max} = 0.2$ rad. Note that the potential energy in the quadratic term changes sign, so the oscillation becomes unsymmetrical, introducing an additional difficulty in finding the equilibrium position.

These conclusions are based on the assumption that the laser's reflected spot is only the size of one pixel. In reality, it spans several pixels and the wiimote uses a centering algorithm to locate the spot and record its position to one pixel at its center. Despite the fact that this is an approximation, when using the fit curves to compute the error in the resolution, very similar results are found with only a slight decrease in the value for the uncertainty, σ .

The conclusion to this qualitative discussion of the effect of the quadratic term is that the amplitude of the oscillations should be kept less than 0.2 rad and we find in practice that keeping it less than half that gives good results. If a fitting program is used, several points contribute to the determination of a max or min, so we should expect a somewhat more stringent limit than we obtained from attributing the error to a single point in the discussion above.

7 References

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