Wave Divisor Function

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Introduction.

The divisor function counts the number of divisors of an integer. A model is described where the divisor function is seen as summation of repeating continuous waves. The divisor function now has a real and imaginary component. This divisor wave model introduces an error in the solution. The wave divisor function method is presented, also a description of the error is given.

1. Describing the divisor function with waves.

The integer divisor function σ_0 [4] can be described as a summation of repeating waves. Each wave filters out numbers. Divisor wave $\mathbb{X} = 7$ wil filter: 7, 14, 21, 28, 35 etc. The divisor function can be described as:

$$\sigma_0(x) = \sum_{\mathbb{X}=2}^{\infty} \cos^N\left(\frac{\pi}{\mathbb{X}}x\right) \tag{1}$$

Here from x the number of divisors is determined excluding divisor 1. N should be a positive even integer; only then positive pulses occur so $N \in 2\mathbb{N}$. If: $N \to \infty$ discrete pulses with magnitude 1 occur on the intervals determined by: $\mathbb{X}$. This definition of the divisor function σ_0 does not take 1 in account, for the conventional definition 1 should be added to the wave divisor function. With Euler's formula and the binomial theorem, the function can be rewritten as:

$$\sigma_0(x) = \sum_{\mathbb{X}=2}^{\infty} e^{i\left(\frac{N\pi}{\mathbb{X}}x\right)} 2^{-N} \sum_{k=0}^{N} \binom{N}{k} e^{-i\left(\frac{\pi}{\mathbb{X}}kx\right)} \tag{2}$$

The solution for the divisor function occurs when the angular component is 0 only then pulses of magnitude 1 occur. For the divisor function we can set:

$$e^{i\left(\frac{N\pi}{\mathbb{X}}x\right)} = 1 \tag{3}$$

While $N\pi$ will always be a multiple of 2π because N must be a positive even integer. So, the "Wave Divisor Function" becomes:

$$\sigma_0(x) = \sum_{\mathbb{X}=2}^{\infty} 2^{-N} \sum_{k=0}^{N} \binom{N}{k} e^{-i\left(\frac{\pi}{\mathbb{X}}kx\right)} \tag{4}$$

The n choose k notation (4) can be written in a trigonometric formulation. This notation was found by plotting (4) and analyzed the function. The "Wave Divisor Function" has a Real and Imaginary solution. The Real solution holds the divisor count.

$$\Re(\sigma_0(x)) = \sum_{\mathbb{X}=2}^{\infty} cos^N\left(\frac{\pi}{\mathbb{X}}x\right) \cdot cos\left(\frac{N\pi}{\mathbb{X}}x\right) \tag{5}$$

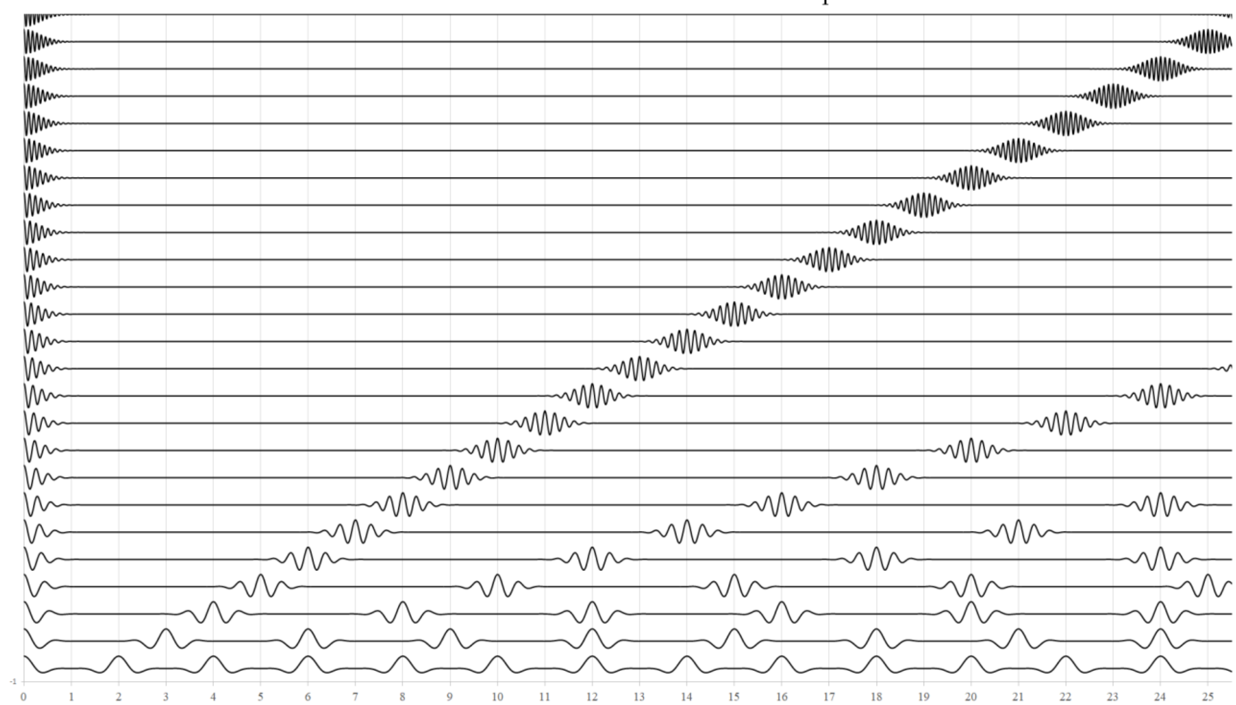
$$\Im(\sigma_0(x)) = -i \sum_{\mathbb{X}=2}^{\infty} cos^N\left(\frac{\pi}{\mathbb{X}}x\right) \cdot sin\left(\frac{N\pi}{\mathbb{X}}x\right) \tag{6}$$

Equations (5) and (6) can be validated by substituting (1) and (2). The following criteria was found:

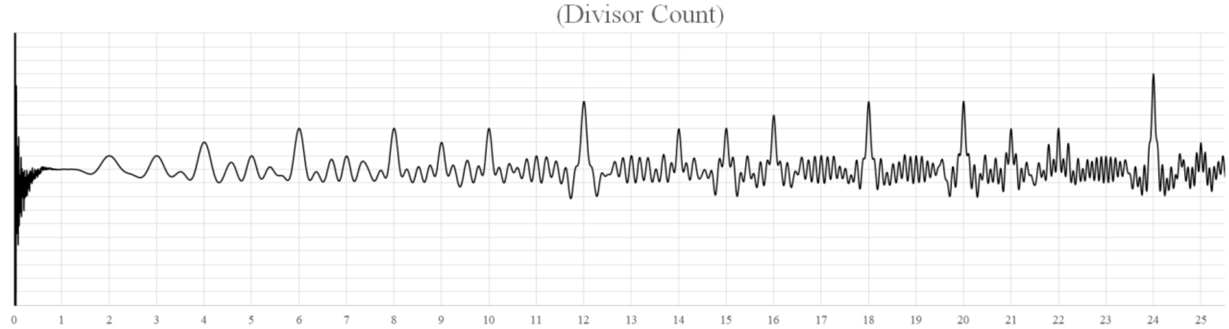
$$cos^2\left(\frac{N\pi}{\mathbb{X}}x\right) + sin^2\left(\frac{N\pi}{\mathbb{X}}x\right) = 1 \tag{7}$$

Thus, the solution of the divisor function is only valid for integer values of x. The wave divisor function consists of repeating wave packages with different frequencies. A wave pulse outline is modulated with a high frequency. When N increases in size the wave packages become narrower and the frequency of the signal increases. One can select a N for every value of $\mathbb{X}$ such that the pulse width for all waves becomes similar.

Individual Divisor Waves Graph

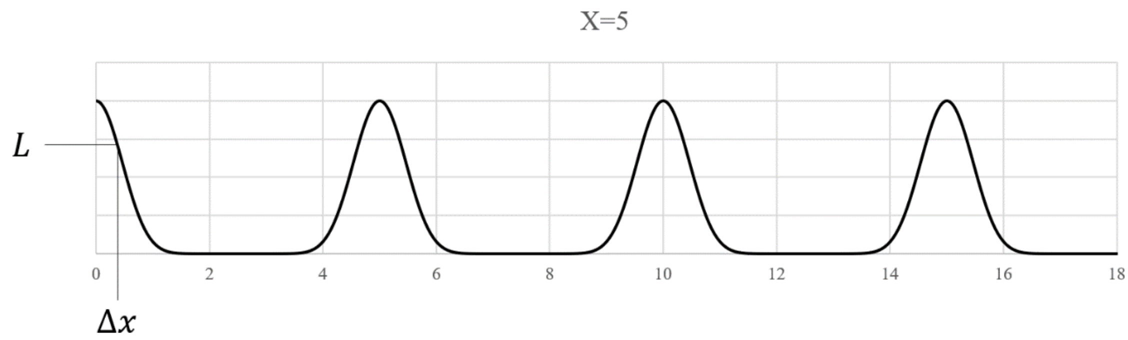


Divisor Wave Summation
(Divisor Count)



2. N, the pulse width definition.

The wave divisor function consists of repeating wave packages defined by cos^N (5). The width of a wave package depends on the size of N. One can define the width around the origin, with the pulse height L at Δx.



$$L = cos^N\left(\frac{\pi}{\mathbb{X}}\Delta x\right) \tag{8}$$

From equation (8) we can calculate the magnitude of N. The wave package width will also vary depending upon the value of $\mathbb{X}$. Thus, N is a function of $\mathbb{X}$. $N(\mathbb{X})$ can derived from (8):

$$N(\mathbb{X}) = \frac{ln(L)}{ln\left(cos\left(\frac{\pi}{\mathbb{X}}\Delta x\right)\right)} \tag{9}$$

Note that $N(\mathbb{X})$ should be an even number $N \in 2\mathbb{N}$, if not negative pulses can occur. Rounding to its closest even number also has a randomizing effect. With help of *Wolfram Alpha* [1] $N(\mathbb{X})$ can also be written as a Taylor series.

$$N(\mathbb{X}) = -\frac{2\mathbb{X}^2\, log(L)}{\pi^2 \Delta x^2} + \frac{log(L)}{3} + \mathcal{O}\left(\frac{1}{\mathbb{X}^2}\right) \tag{10}$$

Typically, in simulations $0 < L \leq 0.5$ and $0 < \Delta x \leq 0.5$ is picked. Though every pulse width setting has multiple combinations of L and Δx. Some investigation on $N(\mathbb{X})$ and a mirror point where $\Delta x \to 1$ is found in: [cc].

3. The Pulse Outline and High Frequency Component.

The wave divisor function consists of a pulse outline $O(x)$ modulated with a high frequency component. It is found that the pulse outline $O(x)$ reaches a limit when $\mathbb{X} \to \infty$. There is a solution to the following limit, note that for N eq. (9) must be substituted.

$$O(x) = \lim_{\mathbb{X}\to\infty} cos^N\left(\frac{\pi}{\mathbb{X}}x\right) = e^{ax^2} \tag{11}$$

$$a = \frac{\log(L)}{\Delta x^2} = constant$$

The solution was found with help of *Wolfram Alpha* [1]. For a given pulse width defined by L and Δx the outline will tend to a bell-shaped curve around the origin for $\mathbb{X} \to \infty$. For this limit $N \to \infty$ when $\mathbb{X} \to \infty$. The limit (11) will hold independent of the fact that N is a positive even integer, at ∞ parity is neglectable.

The wave packages are modulated (5) with a high frequency component $HF(x)$. When $\mathbb{X} \rightarrow \infty$ a limit is found in $HF(x)$. In the following expression N should be substituted with eq. (9).

$$HF(x) = \lim_{\mathbb{X}\rightarrow\infty} cos\left(\frac{N\pi}{\mathbb{X}}x\right) \approx \cos{(bx)} \tag{12}$$

$$b(\mathbb{X}) = \frac{N}{\mathbb{X}}\pi \approx -\frac{2\,log(L)}{\pi\Delta x^2}\mathbb{X} = constant \cdot \mathbb{X}$$

This solution was found with help of eq. (10). It is found that the frequency almost scales linear with $\mathbb{X}$ for narrow pulsewidths.

4. Error in The Wave Divisor Function.

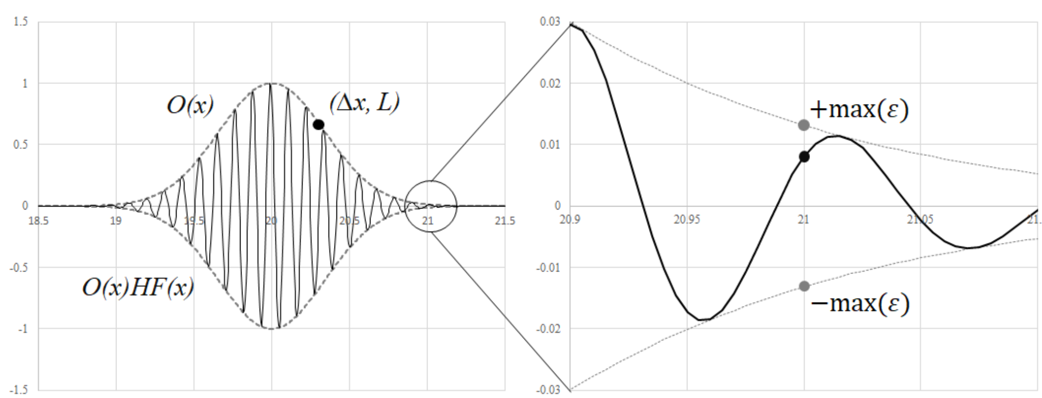
The error of the wave divisor function is majorly determined by neighbor pulses like: $\sigma_0(x-1)$ and $\sigma_0(x+1)$. On these coordinates the value of the function is not zero. The maximum error from a direct neighbor can be determined from the wave pulse outline eq. (11) for x=1:

$$max(\varepsilon) = exp\left(\frac{log(L)}{\Delta x^2}\right) \tag{13}$$

Also $\sigma_0(x-m)$ and $\sigma_0(x+m)$ contribute to the error. For pulses m steps away, we can determine the maximum error from the wave pulse outline eq. (11) for x=m:

$$\varepsilon(m) = exp\left(\frac{log(L)}{\Delta x^2}m^2\right) \tag{14}$$

Error by Individual Divisor Wave



In between the limits defined by eq. (13) and (14) the error will occur. The exact value of the error is determined by $HF(x)$ eq. (12). The frequency of $HF(x)$ scales almost linear with $\mathbb{X}$. For direct neighbor divisors the error can be formulated with eq. (15) where $\mathbb{X}|(x-1)$ means $\mathbb{X}$ divides $(x-1)$, k is a constant determined by the pulse width eq. (12).

$$\varepsilon(x) \approx max(\varepsilon)\cdot\left[\sum_{\mathbb{X}|(x-1)} cos(k\mathbb{X}) + \sum_{\mathbb{X}|(x+1)} cos(k\mathbb{X})\right] \tag{15}$$

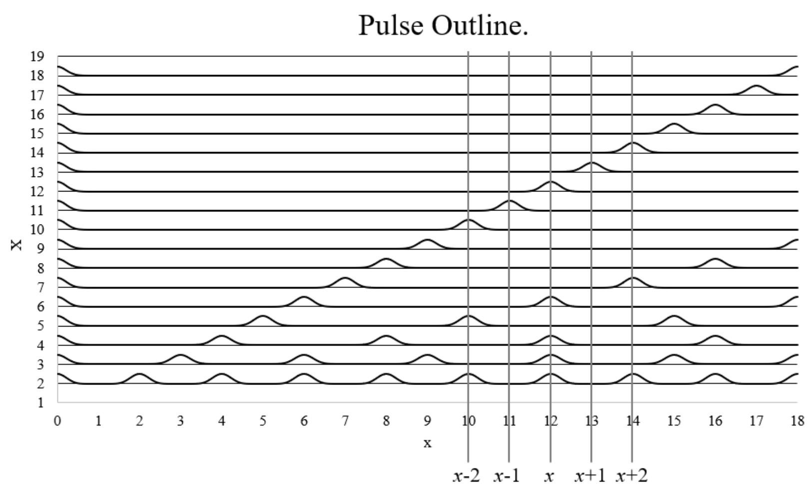
It is assumed that for large values x its divisors are randomly distributed. Also, the rounding of N to its closest even integer causes a randomizing effect, using eq. (12). It is expected that the error is picked from an arcsine distribution. A simulation of the arcsine distribution is available in [ab]. The Variance in the case of an arcsine distribution can be calculated [2] [ab]. For neighbor pulses at $(x-1)$ and $(x+1)$ the variance is:

$$Var(x \pm 1) = \frac{1}{2} \cdot max^2(\varepsilon) \tag{16}$$

Variance for pulses m steps away is:

$$Var(x \pm m) = \frac{1}{2} \cdot \varepsilon^2(m) \tag{17}$$

The total error will be the sum of errors from direct neighbor divisors: $\sigma_0(x-1)$ and $\sigma_0(x+1)$. Also, the error of divisors m steps away must be added. This summation of errors is related to a random walk [3]. The total variation is the sum of all variations of neighbor pulses/divisors and divisors m steps away from x.



$$Var(x) = \frac{1}{2} max^2(\varepsilon) \left(\sum_{m=1}^{\infty} \frac{\sigma_0(x+m) \cdot \varepsilon^2(m)}{max^2(\varepsilon)} + \sum_{m=1}^{\infty} \frac{\sigma_0(x-m) \cdot \varepsilon^2(m)}{max^2(\varepsilon)} \right) \tag{18}$$

The error description of eq. (18) is not ideal. Errors m steps away can be counted duplet, like divisor of $\mathbb{X} = 2$ could be counted double. Though, when the pulse width is small $\Delta x \to 0$ the error converges. The error will be determined by direct neighbor divisors (19). Thus, counting duplets is not the case for $\Delta x \to 0$. This relation takes a sort of mean value of the divisor count (20).

$$Var(x) = \frac{1}{2} max^2(\varepsilon) \left(\sigma_0(x+1) + \sigma_0(x-1) \right) \tag{19}$$

$$Var(x) \approx max^2(\varepsilon) \cdot \overline{\sigma_0(x)} \tag{20}$$

The mean divisor growth is given by Dirichlet [4]. The error term $\mathcal{O}(x^{\Theta})$ in the Dirichlet mean divisor count is not introduced in this writing. Note that an extra (-1) is added, the Wave Divisor Function is excluding divisor: 1.

$$\overline{D(x)} \approx log(x) + 2\gamma - 1 - (1) \tag{21}$$

The Standard Deviation in the Wave Divisor Function can then be approximated with:

$$Stdev(x) \approx max(\varepsilon) \cdot \sqrt{log(x) + 2\gamma - 2} \tag{22}$$

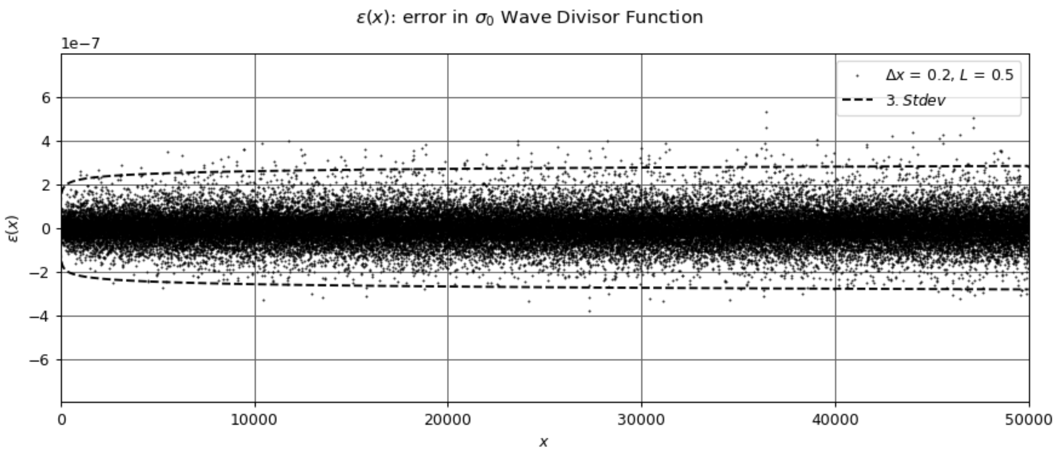
5. Simulation of the Error.

For a given pulse width e.g. $L = 0.5, \Delta x = 0.2$ the divisor count can be determined. The error in the Wave Divisor can be calculated as:

$$\varepsilon(x) = \sigma_0(x)_{Wave} - \sigma_0(x)_{Discrete} \tag{23}$$

The error is calculated for all integers x till the number *50000* in the presented simulation. The boundaries determined by eq. (22) have been plotted as *3Stdev (99.7%)*. Several observations can be made:

1) There occur more positive errors. Analysis showed that more positive errors occur for odd $x's$. This is related to the parity of neighbor divisors of x. Odd numbers only have odd divisors and have symmetrical error distributions. Even numbers have even and odd divisors, the errors for even numbers have non-symmetrical skewed distribution. More information [aa].
2) For $L = 0.5, \Delta x = 0.2$ and integers x till the number *50000*, *99.606%* is counted within the *3Stdev (99.7%)* boundaries. The difference cannot be explained, possible: skewed error distributions and the error term $\mathcal{O}(x^{\Theta})$ in Dirichlet's (21) divisor counting function are involved.



6. More information and Simulations.

The wave divisor function can also be expressed to determine higher order solutions of the divisors function: $\sigma_w(x)$. The wave divisor function will look like:

$$\sigma_w(x) = \sum_{X=2}^{\infty} \mathbb{X}^w \, 2^{-N} \sum_{k=0}^{N} \binom{N}{k} e^{-i\left(\frac{\pi}{\mathbb{X}}kx\right)} \tag{24}$$

$$\Re(\sigma_w(x)) = \sum_{\mathbb{X}=2}^{\infty} \mathbb{X}^w cos^N\left(\frac{\pi}{\mathbb{X}}x\right) \cdot cos\left(\frac{N\pi}{\mathbb{X}}x\right) \tag{25}$$

$$\Im(\sigma_w(x)) = -i\sum_{\mathbb{X}=2}^{\infty} \mathbb{X}^w cos^N\left(\frac{\pi}{\mathbb{X}}x\right) \cdot sin\left(\frac{N\pi}{\mathbb{X}}x\right) \tag{26}$$

The error in the higher order Wave Divisor Function have not been determined yet. More properties of the Wave Divisor Function have been investigated like: Fourier Transform and Arcsine distribution analysis they can be found in [ab].

Summary and Conclusion.

The divisor function can be expressed as a continuous wave function where wave pulses are modulated with a high frequency. The Wave Divisor Function will then have a Real and Imaginary solution. There will be an error in the solution, the first approximation of the error shows that it grows moderate proportional to $\sqrt{\log{(x)}}$. The absolute error is dependent upon the width of the pulses.

The pulse width can be defined infinite ways by varying L and Δx. There are infinite many Wave Divisor Functions all with different errors.

There are more properties of the Wave Divisor Function not mentioned. Some of these properties can be found in [ab]. Here [ca] one can find links with historic documents and the first attempts understanding the Wave Divisor Function. The authors background in Math / Number Theory is too limited to draw any further conclusions.

Questions and Open Issues.

1) Is the error description derived from the trigonometric notation valid Eq. (5)? Eq. (7) states that the trigonometric notation is only valid for integer values of x.
2) The frequency of the wave pulses keeps increasing for larger numbers. The wave pulse than oscillates rapidly between -1 and 1. A computable solution is than not expected.
3) Can for a given pulse width L and Δx the error be estimated as an arcsine distribution?

More Information Wave Divisor Function:

[aa] Stacks Exchange: Q&A Wave Divisor Function.
https://math.stackexchange.com/q/3427431/650339

[ab] Jupyter Notebook File
https://mybinder.org/v2/gh/oooVincentooo/Shared/master?filepath=Wave%20Divisor%20Function%20rev%202.4.ipynb

Audio Simulation:

[ba] Audio simulation
https://youtu.be/8qtZJ6yp5D0

References:

[1] Wolfram Alpha:
https://www.wolframalpha.com/
a) limit ln(L)*ln(cos(m*pi/X))/ln(cos(pi*Delta/X)) as X->infinity
b) ln(L)/(ln(cos(pi*delta/X))) as X->infinity

[2] Arcsine Distribution:
http://openturns.github.io/openturns/1.9/user_manual/_generated/openturns.Arcsine.html

[3] Random Walk:
https://stats.stackexchange.com/questions/159650/why-does-the-variance-of-the-random-walk-increase

[4] Divisor Function
http://mathworld.wolfram.com/DivisorFunction.html

Older Documents by Author:

[ca] Part II & I: “First explorations”, 2018-2014
https://drive.google.com/open?id=11wQfq6RoR5VJG8kaVQpg0F4WYeIVa7mm

[cb] Part III: “Orbitals”, 2018
https://drive.google.com/open?id=1_NtoCXR1YqWuLZDI_F2IdjLsyZYlCaXX

[cc] Part IV: “Error Divisor model”, 2018
https://drive.google.com/open?id=1WrGmtGHkVqhblBYWwpGp12hd3KkK4MKI

[cd] Part V: “Error Divisor Model Part II” rev 1.5, 2018
https://drive.google.com/open?id=1kvnbefcRrl-ZpPBLBgJ67weOSAP5F8u7

[ce] Part VI: “Divisor function Properties n choose k”, 2019
https://drive.google.com/open?id=1VIfDsxPRnWyOwl_5wjOetjgLudaOE1g1

[cf] Concept Summary, 2019
https://drive.google.com/open?id=1PRdMWuRwXttwrvemogZd_J01dpy8aEtO

[v1] Video 1: n choose k Full scale.
https://youtu.be/sbOjuFmq86s

[v2] Video 2: n choose k Origin.
https://youtu.be/uRL_wDaZTuo

[v3] Video 3: n choose k Zoomed in on origin.
https://youtu.be/9uvhaucBl-g

[v4] Video 4: Orbitals of numbers.
https://youtu.be/fLhLaCf4xcM

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