# Accelerated observers in special relativity

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## 1 Some special relativity

Remember that in special relativity we have different notions of time due to the fact that the speed of light $c$ is the same for all inertial observers. If we consider two events $A$ and $B$ in spacetime, we can distinguish three different 'intervals' between these two events. If we introduce an inertial observer who uses coordinates[1] $\{x^\mu\} = \{ct, x, y, z\}$, the three intervals are the following:

- The elapsed *coordinate time* between the two intervals as measured by our observer: $\Delta t = t_B - t_A$. This interval is, of course, coordinate (and hence observer) dependent.
- The elapsed *proper time* as measured by another observer who travels between the two events: $\Delta\tau = \tau_B - \tau_A$. This proper time measures the arc length of the curve which the other observer traverses in spacetime. As such all inertial observers agree about $\Delta\tau$.
- The spacetime interval $\Delta s$, which is the spacetime distance between the two events. It equals the proper time of an *inertial* observer who would travel between the two events and as such would traverse a straight line between the two events.

From now on we will consider two-dimensional spacetime for simplicity. The Lorentz transformations between two inertial observers having a mutual velocity $v$ and using coordinates $\{x^\mu\} = \{ct, x\}$ and $\{x^{'\mu}\} = \{ct', x'\}$ respectively, are

$$\begin{aligned} t' &= \gamma\Big(t - \frac{vx}{c^2}\Big)\,, \\ x' &= \gamma\Big(x - vt\Big)\,, \end{aligned} \tag{1}$$

with $\gamma = \gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. The explicit Lorentz-transformation reads

$$x^{'\mu} = \Lambda^\mu{}_\nu x^\nu\,, \quad \Lambda^\mu{}_\nu = \begin{bmatrix} \gamma & -\frac{\gamma v}{c} \\ -\frac{\gamma v}{c} & \gamma \end{bmatrix} \tag{2}$$

Indeed, one can now check that the spacetime interval $\Delta s$ between two events has the same value for both observers:

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta t')^2 + (\Delta x')^2 = (\Delta s')^2\,. \tag{3}$$

Having different notions of time, we can parametrize wordlines of objects in spacetime with different parameters. We choose to parametrize wordlines with

[1] We will use the terms 'coordinates', 'observers' and 'frames' interchangeably.

the proper time of the object for convenience. Imagine that the worldline of a particle moving with constant velocity with respect to a frame $\{x^\mu\} = \{ct, x\}$ is measured in that frame. This means that we can write this worldline as

$$x^\mu(\tau) = \Big(ct(\tau), x(\tau)\Big)\,, \tag{4}$$

where $\tau$ is the proper time of the particle. Now we transform to the rest frame of the particle, whose coordinate we denote as $\{x^{'\mu}\} = \{ct', x'\}$. The spacetime interval between two events is then

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta t')^2\,. \tag{5}$$

After all, being in the rest frame of a particle means per definition that $\Delta x' = 0$ between two events! Moreover, being in the rest frame of the particle also means that $t'$ is just the proper time $\tau$ of the particle. Hence

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta \tau)^2\,. \tag{6}$$

Being a scalar equation, all inertial observers will agree upon this, which motivates the definition

$$(\Delta s)^2 = -c^2(\Delta \tau)^2\,. \tag{7}$$

Now consider a particle moving along a general (i.e. possibly curved) trajectory, viewed from an inertial frame $\{x^\mu\} = \{ct, x\}$. The trajectory of the particle is described by the wordline $\Big(ct(\tau), x(\tau)\Big)$. Then infinitesimally,

$$d\tau = \sqrt{(dt)^2 - \frac{(dx)^2}{c^2}} = dt\sqrt{1 - \frac{u^2}{c^2}} \quad \rightarrow \quad \frac{dt}{d\tau} = \gamma(u)\,. \tag{8}$$

Note that in $\gamma(u)$, the speed $u = \frac{dx}{dt}$ does not need to be constant! The fact that time dilation only depends on the velocity $u(t)$, and not on the acceleration $\frac{du}{dt}$, is called the *clock hypothesis*.

Now we will define the four-velocity of the particle:[2]

$$U^\mu = \frac{dx^\mu}{d\tau} = \Big(c\frac{dt}{d\tau}, \frac{dx}{d\tau}\Big)\,. \tag{9}$$

But using eqn.(8) and the chain rule, this four-velocity can also be written as

$$U^\mu = \frac{dt}{d\tau}(c, \frac{dx}{dt}) = \frac{dt}{d\tau}\Big(c, u\Big) = \gamma\Big(c, u\Big)\,. \tag{10}$$

The four-velocity is, by its definition, a four-vector. After all, we know that under a Lorentz transformation one has

$$x^\mu \rightarrow x^{'\mu} = \Lambda^\mu{}_\nu x^\nu\,, \tag{11}$$

and $\tau$ is Lorentz-invariant. This means that the four-speed $\sqrt{U_\mu U^\mu}$ must also be Lorentz-invariant. Let's calculate this quantity in the rest frame of the particle in question:

$$U^\mu = \Big(c, 0\Big) \quad \rightarrow \quad U_\mu U^\mu = -c^2\,. \tag{12}$$

[2]We denote four-vectors with capitals, and three-vectors with normal letters.

And just as a check: in general one has, from eqn.(10),

$$U_\mu U^\mu = \gamma^2(-c^2+u^2) = \gamma^2 - c^2\Big(1-\frac{u^2}{c^2}\Big) = -c^2\frac{\gamma^2}{\gamma^2} = -c^2\,. \qquad (13)$$

Note that this means that the four-velocity $U^\mu$ has only *three* independent components, just as the three-velocity in Newtonian mechanics!

## 2 Acceleration in Newtonian relativity

Here we will discuss acceleration in Newtonian physics, for comparison with relativistic acceleration later on. In Newtonian mechanics, we can introduce an inertial observer who uses the coordinate frame $\{t, x\}$. The trajectory of a particle is then described in this frame by the wordline

$$x^\mu(\tau) = \Big(t(\tau), x(\tau)\Big)\,. \qquad (14)$$

where $\tau$ is the proper time of the particle. This description seems a bit redundant; after all, in Newtonians spacetime time is absolute and we can *always* set $\tau = t$. But let's stay redundant, to make the comparison between special relativity. Then we can introduce the four-velocity

$$U^\mu = \frac{dx^\mu}{d\tau} = \Big(\frac{dt}{d\tau}, \frac{dx}{d\tau}\Big) \quad \Big( \; = \Big(1, u\Big) \quad \Big)\,, \qquad (15)$$

where we defined $u = \frac{dx}{dt}$. It's clear that this four-velocity has *three* independent components, and that we cannot calculate a norm because we don't have the Minkowski metric at our disposal. Of course we could calculate the three-norm of the velocity, $\sqrt{u^2}$, but this norm takes different values in differential inertial frames. The acceleration, likewise, is defined as

$$A^\mu = \frac{dU^\mu}{d\tau} = \Big(\frac{dU^0}{d\tau}, \frac{dU^1}{d\tau}\Big) \quad \Big( \; = \Big(0, \frac{du}{dt}\Big) \quad \Big)\,. \qquad (16)$$

The trajectory of an accelerated particle with constant acceleration $a$ for which $x(0) = u(0) = 0$, as measured in the frame $\{t, x\}$, can be found by solving Newton's second law:

$$f = m\frac{d^2x}{dt^2} \;\rightarrow\; \frac{d^2x}{dt^2} = \frac{f}{m} = a \;\rightarrow\; x(t) = \frac{1}{2}at^2\,. \qquad (17)$$

Note that $a$ has the same value for *all* inertial observers! Also, if two accelerating observers would hold tight a thread between them of fixed length, they must have the very same acceleration $a$ in order for the thread not to break. In special relativity, this turns out to be different. Finally, in a spacetime diagram we would write this trajectory as (taking $t \geq 0$)

$$t(x) = \sqrt{\frac{2x}{a}} = \sqrt{\frac{2}{a}}\sqrt{x}\,, \qquad (18)$$

i.e. the spacetime trajectory would be $t(x) \sim \sqrt{x}$.

# 3 Acceleration in special relativity

Now we go back to Einstein's relativity.

## 3.1 Using coordinate time parameterisation

Again, we define

$$A^\mu = \frac{dU^\mu}{d\tau} = \left(\frac{dU^0}{d\tau}, \frac{dU^1}{d\tau}\right). \tag{19}$$

But if we differentiate eqn.(12) with respect to the proper time, we see that

$$\frac{d}{d\tau}\Big(U_\mu U^\mu\Big) = 2U_\mu A^\mu = 0\,, \tag{20}$$

i.e. the four-velocity and four-acceleration are orthogonal everywhere! From the definitions (19) and (10), we can also see that

$$A^\mu = \frac{d}{d\tau}\Big(\gamma c, \gamma u\Big) = \frac{dt}{d\tau}\frac{d}{dt}\Big(\gamma c, \gamma u\Big) = \gamma\Big(\dot\gamma c, \dot\gamma u + \gamma a\Big) \tag{21}$$

with $u = \frac{dx}{dt}$ and $a = \frac{du}{dt}$ the coordinate velocity and acceleration of the particle as measured in $\{ct, x\}$, respectively. Be aware: the dot indicates differentiation with respect to *coordinate* time $t$. There are two things to note here. First, the four-acceleration depends on both $u$ *and* $a$. Second, note that $u(t)$ is not constant necessarily, and hence $\gamma$ is not constant necessarily. Explicitly,

$$\dot\gamma = \frac{d\gamma}{dt} = \gamma^3\frac{u\cdot a}{c^2}\,. \tag{22}$$

Now suppose we're in an inertial (!) frame $\{ct', x'\}$, in which our accelerating particle is at rest in some instantaneous moment. To phrase it differently: only for one specific moment, our inertial frame coincides with the rest frame of the accelerating particle.[3] This means that the accelerating particle crosses the origin $x' = 0$ at some specific value for the coordinate time $t'$. We could call this specific coordinate time $t'_*$. In this frame, the four-acceleration at that very moment $t'_*$ then equals

$$A'^\mu(t'_*, 0) = \gamma\Big(\dot\gamma' c,\ \dot\gamma u' + \gamma' a'\Big) = \Big(0, a'\Big)\,, \tag{23}$$

because $u'(t'_*) = 0$ and hence $\gamma(u') = 1$ and $\dot\gamma' = 0$ (see eqn.22). Now, $A'^\mu$ and $A^\mu$ are related by a Lorentz-transformation,

$$A^\mu = \gamma\Big(\dot\gamma c,\ \dot\gamma u + \gamma a\Big) = \Lambda^\mu{}_\nu A'^\nu\,, \tag{24}$$

where the unusual prime is because we called the 'our accelerated particle is at rest at some moment in time'-frame $\{ct', x'\}$. Performing the Lorentz-transformation, we find

$$A^\mu = \left(\frac{u\gamma a'}{c},\ \gamma a'\right) = \gamma\Big(\dot\gamma c, \dot\gamma u + \gamma a\Big)\,. \tag{25}$$

[3] Or to phrase it even 'differently differently': we can even define a whole *group* of inertial observers, each having a different velocity, whose wordlines/time-axes are tangent to the wordlines of the accelerated particle at every moment in time $t'$.

From the 0-components of this equation, and plugging in eqn.(22), we read off

$$\gamma\dot{\gamma}c = u\gamma\frac{a'}{c} \;\to\; \gamma^4\frac{ua}{c} = \gamma\frac{ua'}{c} \to\; a' = \gamma^3 a\,. \tag{26}$$

This is how the three-acceleration $a$ is perceived by our 'our accelerated particle is at rest at some moment in time'-observer $\{ct', x'\}$ and inertial observer $\{ct, x\}$.

Now we consider constant acceleration. This means that the acceleration, *as measured in the rest frame of the accelerating particle*, is constant. We called this constant acceleration $a'$, and to emphasize we now take it to be constant we will put

$$a' = \frac{du'}{dt'} = \alpha\,. \tag{27}$$

This means that

$$A_\mu A^\mu = \alpha^2\,. \tag{28}$$

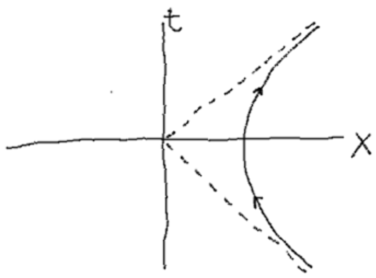


Figure 1: *The wordline of the accelerated particle in the frame* $\{ct, x\}$.

In the frame $\{ct, x\}$ the acceleration $a$ is most certainly not constant; from eqn.(26) we see that

$$a = \gamma^{-3}\alpha = \Big(1 - \frac{u^2}{c^2}\Big)^{3/2}\alpha\,. \tag{29}$$

So, to find the trajectory $x(t)$ of the acceleration particle we can solve for this differential equation by separation of variables with the boundary condition $u(t=0) = 0$:

$$a = \frac{du}{dt} = \Big(1 - \frac{u^2}{c^2}\Big)^{3/2}\alpha \;\to\; u = \frac{\alpha c t}{\sqrt{c^2 + \alpha^2 t^2}}\,. \tag{30}$$

Integrating it once more, such that $x(t=0) = 0$, we find

$$x(t) = \frac{c}{\alpha}\Big(\sqrt{c^2 + \alpha^2 t^2} - c\Big) = \frac{c^2}{\alpha}\Big(\sqrt{1 + \frac{\alpha^2 t^2}{c^2}} - 1\Big)\,. \tag{31}$$

Now a non-relativistic check: if $\alpha$ is so small that the resulting velocity does not approach the speed of light $c$, i.e. $\alpha t << c$, we can make the approximation

$$x(t) \approx \frac{c^2}{\alpha}(1 + \frac{\alpha^2 t^2}{2c^2} - 1\Big) = \frac{1}{2}\alpha t^2\,. \tag{32}$$

This is just eqn.(17), where in this non-relativistic limit $a' = \alpha$ is the same as $a$. The wordline of the particle is sketched in fig.1.

### 3.2 Newton's law . . . relativistically

Could we have found eqn.(31) in another way? Yes, we could: by Newton's law. Newton's law relativistically states that for a particle with worldline $x(t)$, velocity $u(t) = \frac{dx}{dt}$ and acceleration $a = \frac{du}{dt}$,

$$\frac{dp}{dt} = f\,. \tag{33}$$

But now $p = \gamma(u)mu$! Note that $p$ and $f$ are *three*-vectors. If we now again impose a constant proper acceleration $\alpha$ (upon all inertial observers agree), then

$$\frac{d(\gamma(u)mu)}{dt} = m\alpha \ \rightarrow \quad \frac{d(\gamma(u)u)}{dt} = \alpha \quad \rightarrow d\Big(\frac{u}{\sqrt{1-\left(\frac{u}{c}\right)^2}}\Big) = \alpha dt\,. \qquad (34)$$

Again we find, after integration, that

$$u(t) = \frac{c}{\sqrt{1+\left(\frac{c}{\alpha t}\right)^2}} \qquad (35)$$

and also

$$\gamma(u) = \frac{1}{\sqrt{1-\left(\frac{u}{c}\right)^2}} = \sqrt{1+\left(\frac{\alpha t}{c}\right)^2}\,. \qquad (36)$$

## 3.3 Using proper time parameterisation

Could we have found eqn.(31) in *yet* another way? Again: yes, we could. Remember that in the rest frame of the accelerating particle,

$$U^{'\mu} = \Big(c,\ 0\Big)\,. \qquad (37)$$

And because $U'_\mu A^{'\mu} = 0$, we can solve for $A^{'\mu}$:

$$A^{'\mu} = \Big(0,\ \alpha\Big)\,, \qquad (38)$$

such that $A_\mu A^\mu = \alpha^2$. Now we have the following three coupled (differential) equations for $A^\mu = \frac{dV^\mu}{d\tau}$ and $V^\mu$:

$$\begin{aligned} -(V^0)^2 + (V^1)^2 &= -c^2\,, \\ -A^0V^0 + A^1V^1 &= 0\,, \\ -(A^0)^2 + (A^1)^2 &= \alpha^2\,. \end{aligned} \qquad (39)$$

To solve them for $A^\mu$ and $V^\mu$, we rewrite the first equation as

$$(\frac{\alpha}{c}V^0)^2 - (\frac{\alpha}{c}V^1)^2 = \alpha^2 = (A^1)^2 - (A^0)^2 \qquad (40)$$

Hence we find $\frac{\alpha}{c}V^0 = \pm A^1$ and $\frac{\alpha}{c}V^1 = \pm A^0$. Using the very definition of $A^\mu$,

$$\frac{dV^0}{d\tau} = \pm\frac{c}{\alpha}V^1\,, \qquad \frac{dV^1}{d\tau} = \pm\frac{c}{\alpha}V^0\,. \qquad (41)$$

So,

$$\frac{d^2V^0}{d\tau^2} = \frac{\alpha^2}{c^2}V^0\,, \qquad \frac{d^2V^1}{d\tau^2} = \frac{\alpha^2}{c^2}V^1\,. \qquad (42)$$

This equation is suplemented by

$$A^0V^0 = A^1V^1, \quad (V^0)^2 - (V^1)^2 = c^2\,. \qquad (43)$$

One solution to these equations is[4]

$$t(\tau) = \frac{c}{\alpha}\sinh\left(\frac{\alpha}{c}\tau\right), \quad x(\tau) = \frac{c^2}{\alpha}\cosh\left(\frac{\alpha}{c}\tau\right), \tag{44}$$

where $t(\tau = 0) = 0$ and $x(\tau = 0) = \frac{c^2}{\alpha}$. Note that we found the wordline of the particle in terms of its proper time $\tau$, as opposed to the trajectory (31). Of course, these two solutions should be consistent with each-other, as one can check they are. Namely, the elapsed proper time of the accelerated particle is

$$\tau = \int \frac{dt}{\gamma} = \int \sqrt{1 - \frac{u^2(t)}{c^2}}\, dt\,. \tag{45}$$

We can perform this integral, because we have our expression for $u(t)$, namely eqn.(30)! So,plugging eqn.(30) into the integrand we get (see also (36))

$$\sqrt{1 - \frac{u^2(t)}{c^2}} = \sqrt{1 - \frac{\alpha^2 c^2 t^2}{c^2(c^2 + \alpha^2 t^2})} = \frac{1}{\sqrt{1 + \left(\frac{\alpha t}{c}\right)^2}}\,. \tag{46}$$

Performing the integration finally gives, as expected,

$$\tau = \frac{c}{\alpha}\sinh^{-1}\left(\frac{\alpha t}{c}\right), \tag{47}$$

which can be compared to eqn.(44).

The transformation law for the three-acceleration can also be obtained from the usual velocity transformation law of special relativity. Let's recall its derivation. Again, we have two observers using coordinates $\{ct, x\}$ and $\{ct', x'\}$ respectively, which both parametrize the trajectory of a particle by its proper time $\tau$. Both descriptions of the trajectory are related by the Lorentz transformation (2). The observers have a mutual *constant* speed $v$! Now, our particle has a speed $u$ as measured by the observer with coordinates $\{ct, x\}$:

$$\frac{dx}{dt} = u\,. \tag{48}$$

What is the velocity $v'$ as measured by the frame $\{ct', x'\}$? Well,

$$\frac{dx'}{dt'} = u'\,, \tag{49}$$

and by eqn.(2) and remembering that $dv = 0$,

$$\frac{dx'}{dt'} = \frac{\gamma\Big(dx - v dt\Big)}{\gamma\Big(dt - \frac{v}{c^2}dx\Big)} = \frac{\Big(\frac{dx}{dt} - v\Big)}{\Big(1 - \frac{v}{c^2}\frac{dx}{dt}\Big)} = \frac{u - v}{1 - \frac{uv}{c^2}}\,. \tag{50}$$

So we obtain[5]

$$u' = \frac{u - v}{1 - \frac{uv}{c^2}}\,. \tag{51}$$

[4] Using $\cosh^2 - \sinh^2 = 1$.

[5] And also by the product rule $du' = \gamma^{-2} du$.

Now we can go further, and consider the acceleration of the particle in both frames. In the frame $\{ct, x\}$ the three-acceleration is simply $a = \frac{du}{dt}$. In the frame $\{ct', x'\}$ this becomes, due to (51),

$$a' = \frac{du'}{dt'} = \frac{\gamma^2 du}{\gamma\left(dt - \frac{v}{c^2}dx\right)} = \frac{\gamma}{\left(1 - \frac{uv}{c^2}\right)} a \,. \tag{52}$$

If we now again set instantaneously $v = u$ in this transformation law, then we see again that

$$a' = \gamma^3 a = \frac{d(\gamma u)}{dt} \,, \tag{53}$$

i.e. the transformation-law for how accelerations are perceived by different inertial observers.

## 3.4 Rindler coordinates: motivation

The motion described by eqns.(44) is called *hyperbolic motion*, and for a good reason; the coordinates of the accelerated particle as measured in the inertial frame $\{ct, x\}$ obey

$$(x)^2 - (ct)^2 = \frac{c^4}{\alpha^2} \,. \tag{54}$$

The left hand side of this equation is Lorentz-invariant, and the right hand side also, as it should be; after all, $\alpha$ is the *proper* acceleration of the particle. The question now is: how can we use the coordinates (44) to define an accelerating coordinate system which undergoes constant proper acceleration $\alpha$?

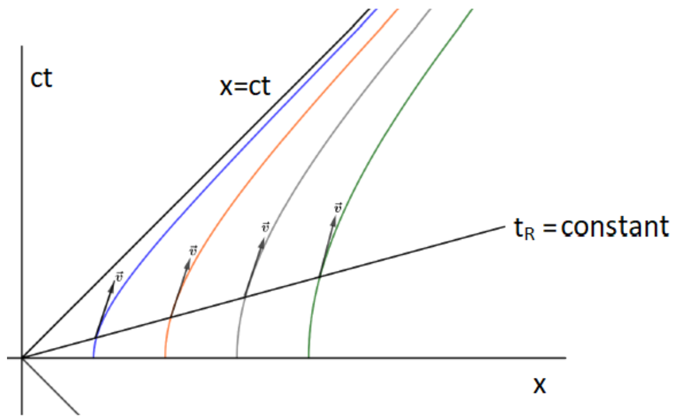


Figure 2: *How the accelerating coordinate grid is defined. Note that for $t_R$ = constant the velocity $v$ is the same at every point $x_R$. Taken (and adjusted) from (J. Kogut).*

Let's name these coordinates first: we call them *Rindler coordinates*. Now we give a qualitative description about how to set up our accelerating coordinate grid, which we will call $\{x^\mu_R\} = \{ct_R, x_R\}$. In the frame of the accelerating observer one has, of course, $t_R = \tau$. She also defines her own position to be constant,[6] $x_R = 0$. Along her acceleration, $x_R$ remains 0. Let's shoot our accelerating observer away with respect to an inertial frame $\{x^\mu\} = \{ct, x\}$. At $t_R = 0$, the accelerating observer sets $\{x_R^\mu\} = \{x^\mu\}$. Now, along her way she synchronizes her clock with those of that particular inertial observer which has, at that very moment, the same velocity as she does. Pictorially, for every value of her proper time, one can draw a tangent line to her wordline; this tangent line defines the time-axis of an inertial observer who has, at that specific moment, the same velocity as she does. The accelerated motion can be seen as a succession of increasing velocities, and her accelerating frame is just the succession of the corresponding inertial frames.[7] For every constant value of $t_R$, the velocities at every $x_R$-value are the same. From this it should be clear that the coordinate lines $t_R =$ constant are straight lines, while the coordinate lines $x_R =$ constant will be curved.

To derive the transformation mathematically, we go back to eqns.(44). We can define the rest frame of the wordline (54) as follows: the time axis is proportional to the tangent vector of the curve, i.e. the four-vector $U^\mu$,

$$U^\mu(\tau) = \frac{dx^\mu}{d\tau} = \Big(\cosh\big(\frac{\alpha}{c}\tau\big), c\sin\big(\frac{\alpha}{c}\tau\big)\ \Big) \quad (U_\mu U^\mu = -c^2)\,. \tag{55}$$

If we choose a vector orthogonal to this $U^\mu$, we get a spacelike vector $S^\mu$. That's easily constructed in two spacetime dimensions:

$$S^\mu(\tau) = \Big(\sinh\big(\frac{\alpha}{c}\tau\big)\ , \cosh\big(\frac{\alpha}{c}\tau\big)\Big)\,. \tag{56}$$

You can check that $S_\mu U^\mu = 0$ and $S_\mu S^\mu = 1$, i.e. $S^\mu$ is a *unit* spacelike vector. So $S^\mu$ defines the spatial direction for our accelerating observer. If we now consider the wordline

$$\xi^\mu(\tau) = x^\mu(\tau) + LS^\mu(\tau)\,, \tag{57}$$

for some constant $L$, then this worldline $\xi^\mu(\tau)$ corresponds to a second accelerating observer ahead of the first accelerating observer $x^\mu(\tau)$, in such a way that they both are separated a *constant* (!) distance $L$ *as measured by the first observer.* Notice also that the $\tau$-parameter is the proper time of this *first* observer, and not of the second one (except, of course, when $S^\mu(\tau) = 0$)! The worldline coordinates $\xi^\mu(\tau)$, like $x^\mu(\tau)$, obey a hyperbolic equation similar to eqn.(54), namely

$$\Big(\xi^1\Big)^2 - \Big(\xi^0\Big)^2 = \Big(\frac{c^2 + L\alpha}{\alpha}\Big)^2\,. \tag{58}$$

Because eqn.(54) defines a particle with proper acceleration $\alpha$ as measured from the inertial frame $\{x^\mu\}$, it follows that an inertial observer having the same velocity as the accelerating observer $\{\xi^\mu\}$ a certain moment $t_*$ measures that this accelerating observer undergoes a constant proper acceleration of

$$\alpha' = \frac{\alpha}{1 + \frac{L\alpha}{c^2}}\,, \quad \text{such that} \quad \Big(\xi^1\Big)^2 - \Big(\xi^0\Big)^2 = \frac{c^4}{(a')^2}\,. \tag{59}$$

[6] Later on we will find that another constant is more convenient.

[7] This is how the coordinate transformation (63) at which we will arrive is derived in chapter 3 of Gerard 't Hooft's *Introduction to General Relativity.*

So we see that, to maintain a constant distance $L$ from each-other (as measured by the first observer), both accelerating observers must have *different* accelerations, namely $\alpha$ and $\alpha'$. This is very different from the Newtonian case, where two observers must have the same acceleration in order to maintain a fixed distance between each other. The reason for this difference, of course, is the relativistic effect of time dilation.

Now, the events with spacetime coordinates

$$\frac{c^2}{\alpha}\Big(\sinh\big(\frac{\alpha}{c}\tau\big),\ \cosh\big(\frac{\alpha}{c}\tau\big)\Big) \tag{60}$$

and

$$\Big(\frac{c^2}{\alpha}+L\Big)\Big(\sinh\big(\frac{\alpha}{c}\tau\big),\ \cosh\big(\frac{\alpha}{c}\tau\big)\Big) \tag{61}$$

with respect to the inertial frame $\{x^\mu\}$ are *simultaneous* for the accelerating observer moving along the wordline (44). So now we construct the Rindler coordinates, which will be denoted as $\{x_R^\mu\}$, with respect to the inertial frame $\{x^\mu\}$:

$$\begin{aligned} ct &= \Big(\frac{c^2}{\alpha}+x_R\Big)\sinh\big(\frac{\alpha t_R}{c}\big)\,, \\ x &= \Big(\frac{c^2}{\alpha}+x_R\Big)\cosh\big(\frac{\alpha t_R}{c}\big)\,. \end{aligned} \tag{62}$$

One sees that the spatial origin $x_R = 0$ indeed describes the wordline of an accelerated particle/observer; eqn.(62) is the proper frame attached to this observer. Also, the Rindler origin $(ct_R, x_R) = (0,0)$ corresponds to $(ct, x) = (0, \frac{c^2}{\alpha})$. The corresponding inverse coordinate transformations of eqns.(62) are

$$\begin{aligned} ct_R &= \frac{c^2}{2\alpha}\ln\Big(\frac{x+ct}{x-ct}\Big)\,, \\ x_R &= \sqrt{x^2-c^2t^2}-\frac{c^2}{\alpha}\,. \end{aligned} \tag{63}$$

We perform three checks on these coordinates. First, a Newtonian one:[8] if the velocity $\frac{\alpha t_R}{c} << 1$, then (62) becomes

$$\begin{aligned} ct &= ct_R + \Big(\frac{\alpha}{c}t_R\Big)x_R + \ldots\,, \\ x &= \frac{c^2}{\alpha}+\frac{1}{2}\alpha^2 t_R^2 + \frac{1}{2}\Big(\frac{\alpha}{c}t_R\Big)^2 x_R + \ldots\,, \end{aligned} \tag{64}$$

which reduces to the well-known Newtonian transformation to an accelerated frame[9]

$$\begin{aligned} t &= t_R\,, \\ x &= \frac{c^2}{\alpha}+\frac{1}{2}\alpha^2 t^2\,. \end{aligned} \tag{65}$$

[8]We use the Taylor approximations $\cosh(x) = 1+\frac{x^2}{2}+\ldots$ and $\sinh(x) = x+\frac{x^3}{6}+\ldots$.

[9]Strictly speaking, this also imposes a condition on $x_R$, but we don't consider this constraint.

Second, if we plug in the wordline-coordinates of the accelerating observer itself into (63), we expect the spatial coordinate $x_R$ to be constant and the time coordinate $t_R$ to be the proper time $\tau$. Well, using the defining condition (54), one sees indeed that $x_R = 0$ for the worldine $\{x^\mu(\tau)\}$ of the accelerating observer. The time coordinate $t_R = 0$ can be checked by using the definition of the hyperbolic geometric functions and calculating $x + ct$ and $x - ct$ via eqn.(44),

$$x(\tau) \pm ct(\tau) = \frac{c^2}{\alpha} e^{\pm\alpha\tau/c} \,. \tag{66}$$

From this one can easily deduce that $t_R(\tau) = \tau$. The last check on eqns.(63) is that the coordinate line $t(x)$ with $t_R =$ constant describes a straight line in the coordinate frame $\{ct, x\}$, while the lines $t(x)$ with $x_R =$ constant are curved; this can also be easily checked. These checks motivate that the coordinate frame (63) attached to our accelerated observer has the right form.

Now the Minkowski line-element changes form if we transform to the coordinate $\{x_R^\mu\}$:

$$ds^2 = -c^2 dt^2 + dx^2 = -\Big(1 + \frac{\alpha x_R}{c^2}\Big)^2 c^2 dt_R^2 + dx_R^2 \,. \tag{67}$$

This is exactly the form which would be expected from the equivalence principle, if we compare this expression with that of the Schwarzschild line-element. As a last result, we look at how outgoing light rays propagate in both frames. In our inertial frame $\{x^\mu\}$, that's easy; if we shoot a light ray from the point $x = x_0$, then this light ray traces the curve

$$x(t) = ct + x_0 \,, \quad \text{or} \quad x(t) - ct = x_0 \,. \tag{68}$$

If we want to describe this curve in the Rindler coordinates, we have to use eqn.(62):

$$x - ct = \Big(\frac{c^2}{\alpha} + x_R\Big) e^{-\alpha t_R/c} = x_0 \,. \tag{69}$$

Differentiating this expression with respect to $t_R$ and solving for $\frac{dx_R}{dt_R}$, we get

$$\frac{dx_R}{dt_R} = \Big(1 + \frac{\alpha x_R}{c}\Big) c \,, \tag{70}$$

from which we see that according to the accelerating observer the speed of light depends on the position $x_R$ the light ray has in the accelerating frame. In the accelerating frame, the speed of light *exceeds* $c$ for $x_R > 0$, i.e. when the light ray is ahead of the observer. On the other hand, if the light ray is behind the observer, i.e. $x_R < 0$, then the speed of light is perceived to be *smaller* than $c$. The accelerating observer measures the speed of light to be $c$ only at the origin $x_R = 0$, as expected. If $x_R = -\frac{c^2}{\alpha}$, then the light ray has stopped according to the accelerating observer.[10] The result (70) can also be obtained by putting $ds = 0$ in the interval (67). The fact that an accelerating observer will eventually measure that the speed of light becomes zero can be compared to a stationary observer outside of a black hole, who measures that the speed of light becomes zero at the horizon. Let's see this comparison a bit more in detail.

[10]Note: if the acceleration equals the Earth's gravitational acceleration, $\alpha = 9,8\, m/s^2$, then this distance $x_R$ equals roughly $10^{16}$ kilometer, or roughly one light-year!

### 3.5 Horizons

Let's consider the following setup. In the inertial frame $\{x^\mu\}$ we have a planet at distance $x = D$ from the origin. Also, at $t = 0$, a rocket accelerates with constant proper acceleration $\alpha$ from the origin $(ct, x) = (0, 0)$. The curve this rocket follows is described by (see eqn.(31))

$$x(t) = \frac{c^2}{\alpha}\left(\sqrt{1 + \left(\frac{\alpha t}{c}\right)^2} - 1\right). \tag{71}$$

Somebody on the accelerating rocket measures the distance from his/her rocket to the planet as $\frac{D - x(t)}{\gamma(t)}$ due to length contraction. But if we plug eqn.(36) into this expression, we obtain

$$\frac{\Big(D - x(t)\Big)}{\gamma(t)} = \frac{\Big(D + \frac{c^2}{\alpha}\Big)}{\sqrt{1 + \left(\frac{\alpha t}{c}\right)^2}} - \frac{c^2}{\alpha}. \tag{72}$$

Now, if we send $t_R \to \infty$, this also means that $t \to \infty$ according to eqn.(63). So we can conclude that eventually our accelerating observer will measure that the distance between her and the planet becomes

$$\lim_{t \to \infty} \frac{\Big(D - x(t)\Big)}{\gamma(t)} = -\frac{c^2}{\alpha}. \tag{73}$$

So the planet is now *behind* (because of the minus-sign) the rocket at a distance $-\frac{c^2}{\alpha}$ ... but this result does not depend on the initial distance $D$! So after a very long time $t_R$ the planet will stop receding away from the rocket and 'freeze' at a distance $x_R = -\frac{c^2}{\alpha}$. In fact, *all* objects which initially lie ahead of the rocket will eventually 'freeze' at the distance $x_R = -\frac{c^2}{\alpha}$! The horizon also manifests itself because of the validity of eqn.(63); the coordinates $\{x_R^\mu\}$ are only defined for the wedge bounded by $x = +ct$ and $x = -ct$ (otherwise the arguments of the square root and/or logarithm become negative):

$$-ct < x < ct. \tag{74}$$

The horizon also agrees with our findings after eqn.(70) that the speed of light becomes zero for the accelerating observer at $x_R = -\frac{c^2}{\alpha}$. And lastly, the horizon also follows from eqn.(67): for $x_R = -\frac{c^2}{\alpha}$, we have $g_{00} = 0$, similarly to the vanishing of $g_{00}$ at the Schwarzschild event horizon $r = \frac{2GM}{c^2}$ in the usual spherical coordinates. The Rindler horizon can be seen in figure 1 as the dotted line. If we focus on the $(ct_R, x_R) > (0, 0)$ quadrant, we see that above the dotted line even a *light signal* will never receive the accelerated observer. So all points above this dotted line are causally disconnected from the accelerated observer. This is exactly what a horizon constitutes.

# 4 Used literature


- "General Relativity: Extra Problems and Solutions", John R. Boccio,

- "Acceleration: Relativistic Rocket Dynamics and Accelerating Reference Frames", John Kogut,
- "Uniformly accelerated observers in special relativity", G.F. Torres del Castillo and C.I. Pérez Sánchez,
- "Dynamics and Relativity", David Tong,
- "Introduction to General Relativity", Gerard 't Hooft (Version November 2010).