9.10.2 *Deflection of Light by the Sun*

The phenomenon of *deflection of light in strong gravitational fields* was first signaled by Henry Cavendish (1731–1810) in 1784 (in an unpublished manuscript), then by Johann Georg von Soldner (1776–1833) in 1804, and by Siméon Poisson (1781–1840) in 1833. These calculations were made within the framework of Newtonian mechanics. Einstein was the first to calculate the correct value for the light bending.

Let us consider a light ray propagating in the Sun's gravitational field. Its path, according to general relativity, is a geodesic of the Riemannian space, characterized by $ds = 0$ (the so-called *null geodesic*). Admitting that the trajectory is plane, Schwarzschild's metric (9.128) becomes

$$c^2\left(1-\frac{r_S}{r}\right)dt^2-\frac{dr^2}{1-\frac{r_S}{r}}-r^2d\theta^2=0.$$

We also have

$$c^2\left(1-\frac{r_S}{r}\right)dt=\frac{A}{B}\,r^2\,d\theta,$$

where $A/B = \text{const.}$ (see (9.149)). Eliminating the time from the last two equations, one obtains

$$\left(\frac{d\sigma}{d\theta}\right)^2=r_S\sigma^3-\sigma^2+\frac{A^2}{B^2}\frac{1}{c^2},$$

or, by taking the derivative with respect to θ,

$$\frac{\partial^2 \sigma}{\partial \theta^2} + \sigma = \frac{3r_S}{2}\sigma^2. \tag{9.162}$$

To integrate this equation we shall use the method of successive approximations. As it can be remarked, the effect of gravitation is expressed by the right-hand side, which means that equation $\sigma'' + \sigma = 0$ gives the trajectory of light in the absence of gravitation. The solution of this equation is, obviously, a straight line:

$$\sigma_0 = \frac{1}{R}\cos\theta, \tag{9.163}$$

where σ_0 is the zeroth order approximation, and $1/R$ a constant of integration. To obtain the first-order approximation to the solution, we replace (9.163) into the right-hand side of (9.162) and obtain

$$\frac{\partial^2 \sigma}{\partial \theta^2} + \sigma = \frac{3r_S}{2}\frac{1}{R^2}\cos^2\theta,$$

with the solution

$$\sigma_1 = \frac{r_S}{2R^2}\left(\cos^2\theta + 2\sin^2\theta\right).$$

In the first-order approximation, the solution of Eq. (9.162) therefore is

$$\sigma = \sigma_0 + \sigma_1 = \frac{1}{R}\cos\theta + \frac{r_S}{2R^2}\left(\cos^2\theta + 2\sin^2\theta\right), \tag{9.164}$$

and represents the parametric equation of the trajectory of the light ray in the gravitational field of the Sun.

It is more convenient to use Cartesian coordinates instead of the polar ones. By means of the transformation equations $x = r\cos\theta$, $y = r\sin\theta$, Eq. (9.164) takes the form

$$x = R - \frac{r_S}{2R}\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}}, \tag{9.165}$$

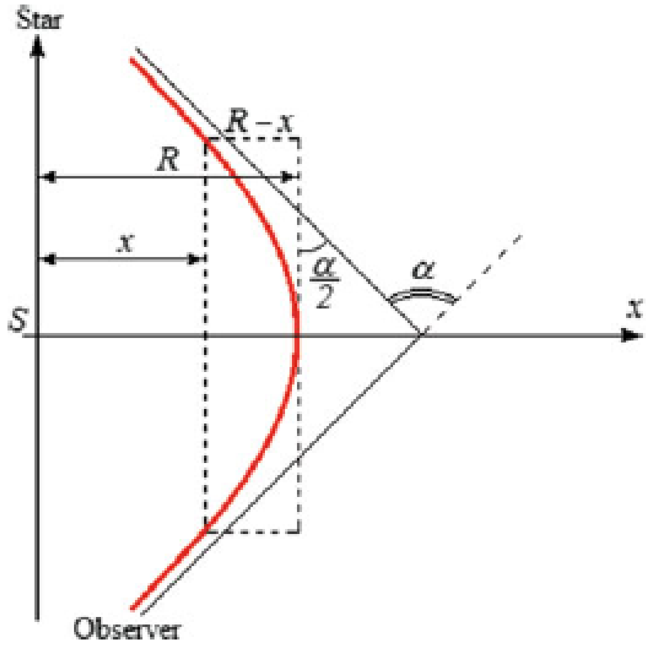
which is a hyperbola, with the Sun at its focus. The value of the deviation is given by the angle between the asymptote to the trajectory and the straight line $x = R$ (see Fig. 9.6), namely

$$\tan\frac{\alpha}{2} = \lim_{y\to\infty}\frac{R - x}{y} = \frac{r_S}{2R}\lim_{y\to\infty}\frac{x^2 + 2y^2}{\sqrt{x^2 + y^2}} = \frac{r_S}{r}.$$

For small angles,

$$\tan\frac{\alpha}{2} \simeq \frac{\alpha}{2} = \frac{r_S}{R},$$

Fig. 9.6 Deflection of a light beam by the Sun.



which yields

$$\alpha \simeq 2\frac{r_S}{R} = \frac{4GM}{c^2R}. \tag{9.166}$$

Approximating R with the radius of the Sun, one obtains $\alpha = 1.75$ arc seconds. The observations performed in 1919 by Arthur Eddington (1822–1944) and his collaborators during a total solar eclipse (in Africa), so that the stars near the Sun could be observed, produced a spectacular confirmation of Einstein's theory. Astronomers now refer to this displacement of light as *gravitational lensing*.

The observations were repeated during other total Solar eclipses over the time. All were in excellent agreement with Einstein's predictions.