0.1 Sample mean

An estimator for the mean value would be:

$$\hat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i \tag{1}$$

The expected value of this estimator:

$$E[\hat{\mu}] = E[\frac{1}{N}\sum_{i=1}^{N} x_i] = \frac{1}{N}\sum_{i=1}^{N} E[x_i] = \mu \tag{2}$$

Since the expected value of $\hat{\mu} = \mu$ this is an unbiased estimator.

$$\begin{aligned} E[\hat{\mu}^2] &= E[(\frac{1}{N}\sum_{i=1}^{N} x_i)^2] = E[\frac{1}{N^2}(\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j)] \\ &= \frac{1}{N^2}E[\sum_{i=1}^{N} x_i^2 + \sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N} x_i x_j] \\ &= \frac{1}{N^2}\sum_{i=1}^{N} E[x_i^2] + \frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1,i\neq j}^{N} E[x_i x_j] = \frac{1}{N}E[x^2] + \frac{N-1}{N}\mu^2 \end{aligned} \tag{3}$$

This leads to the following variance for the estimator:

$$\begin{aligned} \sigma_{\hat{\mu}}^2 &= E[\hat{\mu}^2] - E^2[\hat{\mu}] = \frac{1}{N}E[x^2] + \frac{N-1}{N}\mu^2 - \mu^2 \\ &= \frac{1}{N}E[x^2] - \frac{1}{N}\mu^2 = \frac{E[x^2]-\mu^2}{N} = \frac{\sigma^2}{N} \end{aligned} \tag{4}$$

So:

$$\sigma_{\hat{\mu}} = \frac{\sigma}{\sqrt{N}} \tag{5}$$

0.2 Sample variance

An estimator for the variance using $\hat{\mu}$ would be:

$$\begin{aligned} \hat{\sigma}_N^2 &= \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i^2 - 2x_i\hat{\mu} + \hat{\mu}^2) \\ &= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{2}{N}\sum_{i=1}^{N} x_i\hat{\mu} + \frac{1}{N}\sum_{i=1}^{N}\hat{\mu}^2 \\ &= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{2}{N}\sum_{i=1}^{N}(x_i\frac{1}{N}\sum_{j=1}^{N} x_j) + \frac{1}{N}\sum_{i=1}^{N}(\frac{1}{N}\sum_{j=1}^{N} x_j)^2 \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j + \frac{1}{N^2}(\sum_{i=1}^{N} x_i)^2 \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{2}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j + \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \\
&= \frac{1}{N}\sum_{i=1}^{N} x_i^2 - \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{N} x_i x_j \\
&= \frac{N}{N^2}\sum_{i=1}^{N} x_i^2 - \frac{1}{N^2}\sum_{i=1}^{N} x_i^2 - \frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1, i\neq j}^{N} x_i x_j \\
&= \frac{N-1}{N^2}\sum_{i=1}^{N} x_i^2 - \frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1, i\neq j}^{N} x_i x_j \qquad (6)
\end{aligned}
$$

The expected value of this estimator is:

$$
\begin{aligned}
E[\hat{\sigma}_N^2] &= E[\frac{N-1}{N^2}\sum_{i=1}^{N} x_i^2 - \frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1, i\neq j}^{N} x_i x_j] \\
&= E[\frac{N-1}{N^2}\sum_{i=1}^{N} x_i^2] - E[\frac{1}{N^2}\sum_{j=1}^{N}\sum_{i=1, i\neq j}^{N} x_i x_j] \\
&= \frac{N-1}{N}E[x^2] - \frac{N-1}{N}\mu^2 = \frac{N-1}{N}(E[x^2] - \mu^2) \\
&= \frac{N-1}{N}\sigma^2 \qquad (7)
\end{aligned}
$$

This estimator is therefore not unbiased. The following estimator obviously is (which is the sample variance):

$$
\hat{\sigma}_{N-1}^2 = \frac{N}{N-1}\frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu})^2 \qquad (8)
$$